

§6 Introduction to Determinants

First we introduce some terminology. An $n \times n$ matrix A is *invertible* if the corresponding linear transformation $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ has an inverse T_A^{-1} . The *inverse* A^{-1} of A is the matrix corresponding to T_A^{-1} , i.e. $T_{A^{-1}} = T_A^{-1}$.

The purpose of this lecture is to give an algebraic criterion for when a matrix is invertible. First, we consider the 2×2 case. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

be a 2×2 matrix. In the exercises, we proved the formula

$$\begin{aligned} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} &= \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\ &= (a_{11}a_{22} - a_{12}a_{21}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

From this formula, it is clear that if $a_{11}a_{22} - a_{12}a_{21} \neq 0$ then A is invertible and

$$A^{-1} = \frac{1}{(a_{11}a_{22} - a_{12}a_{21})} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$$

On the other hand, if $a_{11}a_{22} - a_{12}a_{21} = 0$ then the vectors $(a_{22}, -a_{21})$ and $(-a_{21}, a_{11})$ are necessarily in the kernel of A ; this means that A cannot be invertible.

This motivates the definition of the *determinant* of a 2×2 matrix. If

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

then the *determinant* of A , denoted $\det(A)$, is defined by

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

Our next goal is to define the determinant of an $n \times n$ matrix. There are two basic approaches to defining the general determinant: we can either use a recursive formula or we can define it by the properties it should have. We take the first approach.

Before we get started, we need one more definition. Let A be an $n \times n$ matrix. We define the *ij minor* of A , denoted A_{ij} , to be the $(n-1) \times (n-1)$ matrix obtained by removing the *i*th row and the *j*th column of A .

Example Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

Some examples of minors of A are

$$A_{11} = \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} \quad A_{23} = \begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix}.$$

Definition Let A be an $n \times n$ matrix. The determinant of A is defined recursively as follows. If $n = 1$ and $A = (a_{11})$ then $\det(A) = a_{11}$. For general n , we have the formula

$$\det(A) = a_{11} \det(A_{11}) + \dots + (-1)^{1+j} a_{1j} \det(A_{1j}) + \dots + (-1)^{1+n} a_{1n} \det(A_{1n}).$$

This definition is recursive in the sense that the determinant of an $n \times n$ matrix is computed from the determinants of several $(n-1) \times (n-1)$ matrices, the determinants of these $(n-1) \times (n-1)$ matrices may be computed from the determinants of $(n-2) \times (n-2)$ matrices, etc. This process is called expanding a determinant by minors.

To illustrate this, we give formulas for the determinants of small matrices. First, we point out that the recursive formula agrees with the definition given above for 2×2 matrices. Now let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Using the recursive formula, we obtain

$$\begin{aligned} \det(A) &= a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + a_{13} \det(A_{13}) \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

These explicit expressions for the determinant become unmanageable as n becomes large. The expression for the determinant of a general $n \times n$ matrix has $n!$ terms.

Example: Let I_n be the $n \times n$ identity matrix. Applying the recursive formula for the determinant, we find that $\det(I_n) = 1 \det(I_{n-1})$. Since $\det(I_1) = 1$ we conclude that

$$\det(I_n) = 1.$$

Remark: Let (v_1, v_2, v_3) and (w_1, w_2, w_3) be vectors in \mathbb{R}^3 . The *cross product* of v and w can be computed by the determinantal formula

$$v \times w = \det \begin{pmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ e_1 & e_2 & e_3 \end{pmatrix}$$

where e_1, e_2, e_3 are the standard basis vectors. For instance $(1, 0, 0) \times (1, 2, 0) = (0, 0, 2)$.

For the results that follow, we use $a_{*1}, a_{*2}, \dots, a_{*n}$ to denote the column vectors of an $n \times n$ matrix A , i.e. $A = (a_{*1}, a_{*2}, \dots, a_{*n})$. We sometimes think of the determinant as a function of the column vectors.

Theorem Let $A = (a_{*1}, a_{*2}, \dots, a_{*n})$ be an $n \times n$ matrix, b a column vector in \mathbb{R}^n , and C another $n \times n$ matrix. The determinant satisfies the following properties:

I) Antisymmetry Property-Exchanging two columns of A changes the sign of the determinant of A :

$$\det((a_{*1}, \dots, a_{*j}, \dots, a_{*k}, \dots, a_{*j}, \dots, a_{*n})) = -\det((a_{*1}, \dots, a_{*k}, \dots, a_{*j}, \dots, a_{*n}))$$

II) Linearity Property-The determinant is linear in each column:

$$\begin{aligned} & \det((a_{*1}, \dots, a_{*j-1}, ra_{*j}, a_{*j+1}, \dots, a_{*n})) \\ &= r \det((a_{*1}, \dots, a_{*j-1}, a_{*j}, a_{*j+1}, \dots, a_{*n})) \\ & \det((a_{*1}, \dots, a_{*j-1}, a_{*j} + b, a_{*j+1}, \dots, a_{*n})) \\ &= \det((a_{*1}, \dots, a_{*j-1}, a_{*j}, a_{*j+1}, \dots, a_{*n})) \\ &+ \det((a_{*1}, \dots, a_{*j-1}, b, a_{*j+1}, \dots, a_{*n})) \end{aligned}$$

III) Multiplicative Property-

$$\det(AC) = \det(A) \det(C)$$

We shall not prove these properties. Their proofs are fairly cumbersome inductive arguments, and may be found in any linear algebra book.

Remark: Assume that A is a matrix with two equal columns. Then exchanging the two equal columns leaves the matrix unchanged. Using the antisymmetry property, we have

$$\det(A) = -\det(A)$$

which implies that $\det(A) = 0$.

We shall use these properties to deduce the following theorem:

Theorem Let A be an $n \times n$ matrix. Then A is invertible iff $\det(A) \neq 0$.

proof (\Rightarrow) Assume that A is invertible with inverse A^{-1} . We have that $AA^{-1} = I_n$, so by the multiplicative property

$$1 = \det(I_n) = \det(A) \det(A^{-1}).$$

This guarantees that $\det(A) \neq 0$.

(\Leftarrow) Assume that A is not invertible. We shall show that $\det(A) = 0$. By the results from lecture 5, we know that $\text{rank}(A) < n$ so the column vectors $a_{*1}, a_{*2}, \dots, a_{*n}$ of A are linearly dependent. We may write $c_1 a_{*1} + c_2 a_{*2} + \dots + c_n a_{*n} = 0$ with $c_j \neq 0$ for some j . Consequently, we obtain

$$a_{*j} = \frac{-1}{c_j} (c_1 a_{*1} + \dots + c_{j-1} a_{*j-1} + c_{j+1} a_{*j+1} + \dots + c_n a_{*n}).$$

Substituting this into the expression for the determinant and using the lin-

earity property, we obtain

$$\begin{aligned}
 \det(A) &= \det((a_{*1} \dots a_{*j-1} a_{*j} a_{*j+1} \dots a_{*n})) \\
 &= \frac{-c_1}{c_j} \det((a_{*1} \dots a_{*j-1} a_{*1} a_{*j+1} \dots a_{*n})) + \dots \\
 &\quad + \frac{-c_{j-1}}{c_j} \det((a_{*1} \dots a_{*j-1} a_{*j-1} a_{*j+1} \dots a_{*n})) \\
 &\quad + \frac{-c_{j+1}}{c_j} \det((a_{*1} \dots a_{*j-1} a_{*j+1} a_{*j+1} \dots a_{*n})) + \dots \\
 &\quad + \frac{-c_n}{c_j} \det((a_{*1} \dots a_{*j-1} a_{*n} a_{*j+1} \dots a_{*n})).
 \end{aligned}$$

But each of these matrices has two equal columns, so their determinants are all zero by the remark above. We conclude that $\det(A) = 0$. \square

Example: Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 5 \\ 3 & 1 & 7 \end{pmatrix}.$$

The columns of this matrix satisfy the dependence relation

$$2a_{*1} + a_{*2} - a_{*3} = 0$$

so we expect $\det(A)$ to be zero. Expanding by minors, we obtain

$$\begin{aligned}
 \det(A) &= 1 \det(A_{11}) - 2 \det(A_{12}) + 4 \det(A_{13}) \\
 &= 1(2) - 2(-1) + 4(-1) = 0.
 \end{aligned}$$

Exercises 1) Compute the determinants of the following matrices:

$$\begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

2) Prove the Antisymmetry, Linearity, and Multiplicative properties for 2×2 matrices.

3) There are many ways to expand a determinant by minors. Verify the following formulas for a 3×3 matrix A :

a) $\det(A) = a_{11} \det(A_{11}) - a_{21} \det(A_{21}) + a_{31} \det(A_{31})$

b) $\det(A) = -a_{21} \det(A_{21}) + a_{22} \det(A_{22}) - a_{23} \det(A_{23})$

4) Let A and B be $n \times n$ matrices. Is the formula $\det(A+B) = \det(A) + \det(B)$ generally true? Prove or give a counterexample.