# §4 Bases and Dimension

## §4.1 Bases

The concept of a basis is useful in solving both computational and theoretical problems in linear algebra. They allow us to reduce calculations on general vector spaces to calculations on $\mathbb{R}^n$.

**Definition** Let $V$ be a vector space, and let $v_1, v_2, \ldots, v_d$ be vectors which span $V$ and are linearly independent. Then we say that $v_1, v_2, \ldots, v_d$ form a *basis* for $V$.

*Examples:*
1)The standard basis vectors $e_1, e_2, \ldots, e_n$ form a basis for $\mathbb{R}^n$.
2)The polynomials $1, x, x^2, \ldots, x^d$ form a basis for the vector space $\mathcal{P}_d$ of polynomails of degree at most $d$.
3)The space $\mathcal{P}$ of polynomials of all degrees has no *finite* basis. We prove this by contradiction. Let $p_1, p_2, \ldots, p_d$ be a finite basis of $\mathcal{P}$. Let $\delta_i = \deg(p_i)$ for $i = 1, \ldots, d$ and let $M = \max(\delta_1, \delta_2, \ldots, \delta_d)$. Then each of the $p_i$ are contained in $\mathcal{P}_M$, so $p_1, p_2, \ldots, p_d$ cannot possibly span $\mathcal{P}$. Thus we have a contradiction.

One fundamental property of basis is summarized in the following result.
**Theorem** Let $V$ be a vector space. The vectors $v_1, v_2, \ldots, v_d$ form a basis for $V$ iff every vector is $V$ may be written uniquely as a linear combination of these vectors.
*proof* ($\Rightarrow$) Let $v_1, v_2, \ldots, v_d$ form a basis for $V$ and let $v$ be a vector in $V$. Since $v_1, v_2, \ldots, v_d$ span $V$, we may write

$$v = r_1 v_1 + r_2 v_2 + \ldots + r_d v_d$$

for some scalars $r_1, r_2, \ldots, r_d$. Assume there exists another representation

$$v = s_1 v_1 + s_2 v_2 + \ldots s_d v_d$$

for some other scalars $s_1, \ldots, s_d$. Subtracting these two expressions, we obtain

$$(r_1 - s_1)v_1 + (r_2 - s_2)v_2 + \ldots + (r_d - s_d)v_d = 0.$$

By the linear independence, we must have that $r_1 - s_1 = r_2 - s_2 \ldots = r_d - s_d = 0$, so the two representations are the same.
($\Leftarrow$) Assume that every vector of $V$ can be written uniquely as a linear combination of $v_1, v_2, \ldots, v_d$. Of course, this immediately implies that $v_1, v_2, \ldots, v_d$ span $V$. Assume now that $c_1 v_1 + c_2 v_2 + \ldots + c_d v_d = 0$ for some scalars $c_1, c_2, \ldots, c_d$. Since 0 can only be written one way as a linear combination of $v_1, v_2, \ldots, v_d$, we conclude that $c_1 = c_2 = \ldots = c_d = 0$. $\square$

Now we give an algorithm which allows us to compute a basis for the span of a finite set of vectors. Our results are summarized in the following proposition:

**Proposition** Let $V = \mathrm{span}(v_1, v_2, \ldots, v_N)$ be the span of some finite set of vectors. Then some subset of $\{v_1, v_2, \ldots, v_N\}$ forms a basis for $V$.

*proof-algorithm* If $v_1, v_2, \ldots, v_N$ are linearly independent then we are done. Otherwise, we have a dependence relation

$$c_1v_1 + c_2v_2 + \ldots + c_Nv_N = 0$$

where $c_1, c_2, \ldots, c_N$ are not all zero. Choose some $j$ so that $c_j \neq 0$. Dividing out, we obtain

$$v_j = -\big(\frac{c_1}{c_j}v_1 + \ldots + \frac{c_{j-1}}{c_j}v_{j-1} + \frac{c_{j+1}}{c_j}c_{j+1} + \ldots + \frac{c_N}{c_j}v_N\big).$$

Consequently, $v_j \in \operatorname{span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_N)$ and also

$$V = \operatorname{span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_N).$$

We continue eliminating vectors in this way until we obtain a linearly independent set of vectors. Since our original set of vectors was finite, this process terminates after a finite number of steps. □

*Example:* Let $V = \operatorname{span}((1,0,2),(2,1,3),(1,1,1))$. Find a basis for $V$.

First we check whether these vectors are linearly dependent. We find that $(-1)(1,0,2) + (2,1,3) + (-1)(1,1,1) = 0$ so we may write

$$(1,1,1) = (-1)(1,0,2) + (2,1,3).$$

Therefore we can eliminate the third vector and write $V = \operatorname{span}((1,0,2),(2,1,3))$. These two vectors are not parallel, so they are linearly independent and form a basis for $V$. Note that there is nothing special about the third vector; we could have eliminated either of the other two.

### §4.2 Introduction to Dimension

Intuitively, we all know that $\mathbb{R} = \mathbb{R}^1$ is one-dimensional and $\mathbb{R}^2$ is two-dimensional. By analogy, we can convince ourselves that $\mathbb{R}^n$ should be $n$-dimensional for all $n$. Now we introduce an algebraic notion of dimension to make this intuition precise.

We shall give the formal definition once we state an important preliminary theorem:

**Dimension Theorem** Let $V$ be a vector space and let $\{v_1, v_2, \ldots, v_d\}$ and $\{w_1, w_2, \ldots, w_k\}$ both be bases for $V$. Then $k = d$.

Because of this result, it makes sense to discuss *the* number of elements in a basis of $V$. This allows us to make the following definition:

**Definition** The *dimension* of a vector space $V$ is the number of elements in a basis for $V$ and is denoted $\dim(V)$. We say that $V$ is *infinite dimensional* if $V$ has no finite basis. By convention, the vector space $\{0\}$ is zero-dimensional.

*Example:* The vector space $M(2,2)$ of $2 \times 2$ matrices has the following basis:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Consequently, this space has dimension four.

Now we turn to the proof of the dimension theorem. The key idea is summarized in the following lemma:

**Lemma**(*Exchange Trick*) Let $\{w_1, w_2, \dots, w_k\}$ and $\{u_1, u_2, \dots, u_d\}$ be bases for $V$. Then for some $j$, $\{w_j, u_2, \dots, u_d\}$ is also a basis for $V$. In other words, we may exchange $u_1$ with $w_j$ for some $j$.

*proof of the lemma* Since the vectors $u_1, \dots, u_d$ are linearly independent, the results from §4 imply

$$\operatorname{span}(u_2, \dots, u_d) \subsetneq \operatorname{span}(u_1, u_2, \dots, u_d) = V.$$

Since $V = \operatorname{span}(w_1, w_2, \dots, w_k)$, we may choose some $w_j$ which is *not* contained in $\operatorname{span}(u_2, \dots, u_d)$. We claim that the set $\{w_j, u_2, \dots, u_d\}$ is then linearly independent. If $cw_j + c_2u_2 + \dots + c_du_d = 0$ then $c = 0$; otherwise we could write $w_j$ as a linear combination of $u_2, \dots, u_d$. But if $c = 0$ then $c_2 = \dots = c_d = 0$ as well because $u_2, \dots, u_d$ are linearly independent.

Since $w_j \in V = \operatorname{span}(u_1, u_2, \dots, u_d)$ we may write

$$w_j = r_1u_1 + r_2u_2 + \dots + r_du_d.$$

The scalar $r_1$ is necessarily nonzero because $w_j$ is not a linear combination of $u_2, \dots, u_d$. Dividing out and regrouping terms, we have

$$u_1 = \frac{1}{r_1}(w_j - r_2u_2 - \dots - r_du_d)$$

so $\operatorname{span}(w_j, u_2, \dots, u_d) = \operatorname{span}(u_1, u_2, \dots, u_d) = V$. We conclude that the set $\{w_j, u_2, \dots, u_d\}$ is a basis for $V$. □

*proof of the theorem* Let $\{v_1, v_2, \dots, v_d\}$ and $\{w_1, w_2, \dots, w_k\}$ be two bases for $V$; we shall prove that $k = d$. Applying the lemma, we may replace $v_1$ by $w_{j(1)}$ for some $j(1)$ to obtain a new basis $\{v_2, \dots, v_{d-1}, w_{j(1)}\}$. Applying the lemma again, we replace $v_2$ by $w_{j(2)}$ for some $j(2)$. Continuing in this way, we obtain a basis

$$\{w_{j(1)}, w_{j(2)}, \dots, w_{j(d)}\}$$

for $V$. Since $\operatorname{span}(w_{j(1)}, w_{j(2)}, \dots, w_{j(d)}) = \operatorname{span}(w_1, \dots, w_k)$, the results of §4 imply that

$$\{w_{j(1)}, w_{j(2)}, \dots, w_{j(k)}\} = \{w_1, w_2, \dots, w_k\}.$$

In particular, these sets have the same number of elements, so $d = k$. □

Intuitively, we expect that a subspace has a smaller dimension than the space it is contained in. We prove the following theorem:

**Theorem** Let $V$ be a vector space of dimension $d$. If $W$ is a subspace of $V$ then

$$\dim(W) \leq d.$$

To prove the theorem, we use the following proposition. Essentially, this proposition says that every linearly independent set may be extended to a basis.

**Proposition** Let $V$ be a $d$-dimensional vector space and let $u_1, u_2, \ldots, u_m$ be linearly independent vectors in $V$. Then these vectors are contained in some basis of $V$, so $m \leq d$.

*proof of proposition/algorithm* Let $u_1, u_2, \ldots, u_m$ be linearly independent vectors in $V$. If these vectors already form a basis then $m \leq d$ by the previous theorem. Otherwise let $\{v_1, \ldots, v_d\}$ be a basis for $V$. Pick some $j_1$ such that $v_{j_1}$ is not contained in the span of $u_1, \ldots, u_m$. The set $\{u_1, u_2, u_m, v_{j_1}\}$ is then linearly independent. (Why?) If this set spans $V$ then we are done. Otherwise, pick some $j_2$ such that $v_{j_2}$ is not contained in the span of $u_1, \ldots, u_m, v_{j_1}$. Continuing in this way, we eventually obtain a basis for $V$. This process terminates after at most $d$ steps; at worst, we can add all $d$ elements of our basis to obtain a set with span equal to $V$. $\square$

*proof of theorem* Let $W \subset V$ be a subspace of a $d-$dimensional vector space. Pick a nonzero vector $u_1$ from $W$. If $W = \text{span}(u_1)$ then $\dim(W) = 1 \leq d$ by the proposition. Otherwise, pick another vector $u_2 \in W \backslash \text{span}(u_1)$. The vectors $u_1$ and $u_2$ must be independent. If $W = \text{span}(u_1, u_2)$ then $\dim(W) = 2 \leq d$ by the proposition. Otherwise, we pick $u_3 \in W \backslash \text{span}(u_1, u_2)$. We continue in this way, constructing a sequence of linearly independent vectors $u_1, u_2, u_3, \ldots$ in $W$. This sequence cannot go on indefinitely. Indeed the proposition implies that it has at most $d$ elements. Consequently, for some $m \leq d$ we must have that $W = \text{span}(u_1, u_2, \ldots, u_m)$ and so $\dim(W) = m \leq d$. $\square$

**Exercises**

1)Find bases and compute the dimension of the following vector spaces:

a)$M(2,3)$

b)$M(4,2)$

c)$M(m,n)$

2)Let $V = \text{span}(e_1 - e_2, e_2 - e_3, e_3 - e_4, e_4 - e_1) \subset \mathbb{R}^4$. Find a basis for $V$ and compute $\dim(V)$.

3)Consider the vectors $u_1 = e_1 + e_2, u_2 = e_1 + e_4 \in \mathbb{R}^4$.

a)Show that $u_1$ and $u_2$ are linearly independent.

b)Find vectors $u_3$ and $u_4$ such that $u_1, u_2, u_3, u_4$ forms a basis for $\mathbb{R}^4$.

4)Let $V$ be a $d$-dimensional vector space and $W \subset V$ a subspace. Show that $W$ is not infinite dimensional.

5)Let $V$ be a vector space, and let $W_1$ and $W_2$ be finite dimensional subspaces of $V$. Define

$$W_1 + W_2 = \{v \in V : v = w_1 + w_2 \text{ for some } w_1 \in W_1, w_2 \in W_2\}.$$

a)Show that $W_1 \cap W_2$ is a subspace of $V$.

b)Show that $W_1 + W_2$ is a subspace of $V$.

c)Prove the formula

$$\dim(W_1 + W_2) + \dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2).$$

*Hint:* First construct a basis $\{u_1, \ldots u_m\}$ for $W_1 \cap W_2$. Then find bases for $W_1$ and $W_2$ containing this basis. Finally, use all these vectors to construct a basis of $W_1 + W_2$.