§3 Linear Combinations, Spans, and Linear Dependence

§3.1 Kernel and Image

Definition Let $T : V_1 \to V_2$ be a linear transformation. The *kernel* of T is the set of vectors $v \in V_1$ such that $T(v) = 0$. It is denoted $\ker(T)$.

Theorem If $T : V_1 \to V_2$ is a linear transformation then $\ker(T)$ is a subspace of V_1.

proof It suffices to show that $\ker(T)$ is closed uner addition and scalar multiplication. Given $v, w \in \ker(T)$, we have

$$T(v + w) = T(v) + T(w) = 0 + 0 = 0$$

by the linearity of T. So $v + w \in \ker(T)$ and the kernel is closed under addition. If r is any scalar, then we have

$$T(rv) = rT(v) = r0 = 0$$

so $rv \in \ker(T)$. Consequently, the kernel is closed under scalar multiplication. □

We can determine whether a linear transformation is injective just by looking at its kernel:

Theorem A linear tranformation $T : V_1 \to V_2$ is injective iff $\ker(T) = \{0\}$.

proof ($\Rightarrow$) Clearly, if T is injective then $\ker(T) = \{0\}$ because only one point is sent to zero by T.
($\Leftarrow$) Assume $\ker(T) = \{0\}$, so that $T(v) = 0$ implies that $v = 0$. Let $w, w' \in V_1$ be vectors such that $T(w) = T(w')$. By the linearity of T, we have

$$T(w - w') = T(w) - T(w') = 0$$

which implies that $w - w' = 0$. □

Example: Let $\mathcal{P}$ denote the vector space of polynomials in x and let $T : \mathcal{P} \to \mathcal{P}$ be defined by the rule

$$T(p) = \frac{d}{dx}(p).$$

(Check this is actually a linear transformation!) Then $p \in \ker(T)$ iff $\frac{d}{dx}(p) = 0$, which occurs only when p is constant, i.e.

$$\ker(T) = \{a_0 : a_0 \in \mathbb{R}\}.$$

Note that for any polynomial q we have $T(q + a_0) = T(q)$.

Definition Let $T : V_1 \to V_2$ be a linear transformation. The *image* of T is simply the range of the function T, i.e. the set of vectors $v \in V_2$ such that $v = T(v_1)$ for some $v_1 \in V_1$. It is denoted $\operatorname{im}(T)$.

Theorem If $T : V_1 \to V_2$ is a linear transformation then $\operatorname{im}(T)$ is a subspace of V_2.

proof We prove that $\operatorname{im}(T)$ is closed under addition and scalar multiplication. Given $v, w \in \operatorname{im}(T)$, we can find $v_1, w_1 \in V_1$ so that $v = T(v_1)$ and $w = T(w_1)$. But then $T(v_1 + w_1) = T(v_1) + T(w_1) = v + w$, so $v + w \in \operatorname{im}(T)$ and the image is closed under addition. If r is any scalar, then $T(rv_1) = rT(v_1) = rv$. This means $rv \in \text{im}(T)$ so the image is closed under scalar multiplication. $\square$

Remark By definition, $T : V_1 \to V_2$ is surjective iff $\text{im}(T) = V_2$.

Example Let $T : \mathcal{P} \to \mathcal{P}$ be the linear transformation defined above. We prove T is surjective. We must show that for any polynomial $p = a_0 + a_1x + \ldots + a_dx^d$ there exists a polynomial q such that $\frac{d}{dx}q = p$. Setting $q = c + a_0x + \frac{1}{2}a_1x^2 + \ldots + \frac{1}{d+1}a_dx^{d+1}$ where c is some constant, we find that $T(q) = p$.

§3.2 Linear Combinations and Spans

Definition Let $v_1, v_2, \ldots, v_k$ be a finite set of vectors in a vector space V. A vector w is a *linear combination* of $v_1, v_2, \ldots, v_k$ if there exist scalars $r_1, r_2, \ldots, r_k$ such that

$$w = r_1v_1 + r_2v_2 + \ldots + r_kv_k.$$

The *span* of the set $v_1, v_2, \ldots, v_k$ is the set of all vectors which may be written as linear combinations of these vectors, i.e.

$$\text{span}(v_1, \ldots, v_k) = \{r_1v_1 + r_2v_2 + \ldots + r_kv_k : r_1, r_2, \ldots, r_k \in \mathbb{R}\}.$$

Remark: One can also define the span of an *infinite* set $S \subset V$. It is simply the set of all vectors which may be written as a linear combination of a finite number of vectors from S.

Theorem For any vectors $v_1, \ldots, v_k \in V$, $\text{span}(v_1, \ldots, v_k)$ is a subspace of V.

proof We prove this set is closed under addition and scalar multiplication. Let $u, w \in \text{span}(v_1, \ldots, v_k)$. Then there exists scalars $r_1, \ldots, r_k, s_1, \ldots, s_k$ such that

$$u = r_1v_1 + \ldots + r_kv_k \quad w = s_1v_1 + \ldots + s_kv_k.$$

Adding these together, we have

$$u + w = (r_1 + s_1)v_1 + \ldots + (r_k + s_k)v_k$$

so $u + w \in \text{span}(v_1, \ldots, v_k)$. Furthermore, for any scalar $r \in \mathbb{R}$ we have

$$ru = rr_1v_1 + \ldots + rr_kv_k$$

so $ru \in \text{span}(v_1, \ldots, v_k)$. $\square$

If $W \subset V$ is a subspace and $W = \text{span}(v_1, \ldots, v_k)$, then we say that W is spanned by $v_1, \ldots, v_k$. Often, the easiest way to describe a subspace explicitly is to represent as the span of some vectors. For example, let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then the image of T is a subspace of $\mathbb{R}^m$. For any vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we have $T(x) = x_1T(e_1) + \ldots + x_nT(e_n)$ so the image of T is spanned by $T(e_1), \ldots, T(e_n)$. These vectors are just the columns of the matrix for T. We summarize this result in the following proposition:

Proposition Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the corresponding matrix. Then $\mathrm{im}(T)$ is equal to the span of the column vectors of A.

§3.3 Computations Involving Spans

In linear algebra, we often need to answer the following question:

Let $v_1, v_2, \ldots, v_k, w$ be vectors in $\mathbb{R}^n$. Is $w \in \mathrm{Span}(v_1, \ldots, v_k)$?

In other words, do there exist scalars $r_1, \ldots, r_k$ such that $w = r_1 v_1 + r_2 v_2 + \ldots + r_k v_k$? First of all, let's write out the components of all these vectors:

$$v_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} \quad \ldots \quad v_k = \begin{pmatrix} a_{1k} \\ \vdots \\ a_{nk} \end{pmatrix} \quad w = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

Our equation then may be written in matrix form:

$$\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = r_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} + \ldots + r_k \begin{pmatrix} a_{1k} \\ \vdots \\ a_{nk} \end{pmatrix} = \begin{pmatrix} a_{11} & \ldots & a_{1k} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nk} \end{pmatrix} \begin{pmatrix} r_1 \\ \vdots \\ r_k \end{pmatrix}.$$

Unwinding the matrix notation

$$\begin{aligned} b_1 &= a_{11} r_1 + a_{12} r_2 + \ldots + a_{1k} r_k \\ &\vdots \\ b_n &= a_{n1} r_1 + a_{n2} r_2 + \ldots + a_{nk} r_k. \end{aligned}$$

To reiterate, $w \in \mathrm{span}(v_1, \ldots, v_k)$ if and only if this system of linear equations has a solution. (Generally, if a system of linear equations has a solution, we say that the system is *consistent*.)

Example 1: Let $v_1 = (1, 2, 3, 4), v_2 = (2, -1, -1, 0) \in \mathbb{R}^4$, and let $w = (4, 3, 5, 8)$. Is $w \in \mathrm{span}(v_1, v_2)$?

Following the reasoning above, we obtain the system of equations

$$\begin{pmatrix} 4 \\ 3 \\ 5 \\ 8 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 3 & -1 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}.$$

Expanding these out, we obtain

$$4 = r_1 + 2r_2 \quad 3 = 2r_1 - r_2 \quad 5 = 3r_1 - r_2 \quad 8 = 4r_1$$

which has solution $r_1 = 2 \quad r_2 = 1$. Consequently, $w = 2v_1 + v_2$ and $w \in \mathrm{span}(w_1, w_2)$.

Example 2: Let $v_1 = (1, 2, 3, 4), v_2 = (2, -1, -1, 0) \in \mathbb{R}^4$, and let $w = (2, 1, 3, 4)$. Is $w \in \mathrm{span}(v_1, v_2)$?

This time, we obtain the equations

$$2 = r_1 + 2r_2 \quad 1 = 2r_1 - r_2 \quad 3 = 3r_1 - r_2 \quad 4 = 4r_1.$$

We shall show that these have no common solutions, by showing they contradict each other (i.e. they are *inconsistent*). The fourth equation implies that $r_1 = 1$. However, adding the first equation and two times the second equation we obtain

$$4 = (r_1 + 2r_2) + 2(2r_1 - r_2) = 5r_1$$

which implies that $r_1 = \frac{4}{5} \neq 1$, a contradiction.

Example 3: Let $v_1 = (1, 2, 3, 4), v_2 = (2, -1, -1, 0)$ and let $w_1 = (4, 3, 5, 8), w_2 = (3, 1, 2, 4)$. Show that

$$\text{span}(v_1, v_2) = \text{span}(w_1, w_2).$$

We must prove the two inclusions

$$\text{span}(v_1, v_2) \subset \text{span}(w_1, w_2) \quad \text{span}(w_1, w_2) \subset \text{span}(v_1, v_2).$$

To prove the first inclusion, it suffices to show that $v_1, v_2 \in \text{span}(w_1, w_2)$. If we can represent v_1, v_2 as linear combinations of w_1 and w_2, then any linear combination of v_1 and v_2 is a linear combination of w_1 and w_2. Solving the linear equations, we find that

$$v_1 = w_1 - w_2 \quad v_2 = 2w_2 - w_1.$$

This proves $\text{span}(v_1, v_2) \subset \text{span}(w_1, w_2)$. To prove the reverse inclusion, it suffices to show that w_1, w_2 can be represented as linear combinations of v_1 and v_2. After solving the linear equations, we find

$$w_1 = 2v_1 + v_2 \quad w_2 = v_1 + v_2$$

so $\text{span}(w_1, w_2) \subset \text{span}(v_1, v_2)$.

§3.4 Linear Dependence

Definition Let $v_1, v_2, \ldots, v_k$ be vectors in a vector space V. We say these vectors are *linearly dependent* if there exist scalars $c_1, \ldots, c_k$, not all equal to zero, such that

$$c_1v_1 + c_2v_2 + \ldots + c_kv_k = 0.$$

Such an equation is called a *dependence relation.*

The vectors $v_1, v_2, \ldots, v_k$ are said to be *linearly independent* if they are not linearly dependent. In other words, if $c_1v_1 + c_2v_2 + \ldots + c_kv_k = 0$ then $c_1 = c_2 = \ldots = c_k = 0$.

Examples:

1)A set consisting of a single vector v_1 is independent if and only if $v_1 \neq 0$.
2)Two vectors v_1, v_2 are independent if and only if both vectors are nonzero and the vectors are not 'parallel', i.e. $v_1 \neq av_2$ for any scalar $a \in \mathbb{R}$.
3)The standard basis vectors $e_1, e_2, \ldots, e_n$ in $\mathbb{R}^n$ are linearly independent.

Theorem Let $v_1, \ldots, v_k$ be linearly independent vectors. Then for each $N = 1, \ldots, k-1$ we have

$$\text{span}(v_1, v_2, \ldots, v_N) \subsetneq \text{span}(v_1, v_2, \ldots, v_N, v_{N+1}).$$

proof Any vector which is a linear combination of $v_1, v_2, \ldots, v_N$ is also a linear combination of $v_1, v_2, \ldots, v_{N+1}$, so the inclusion

$$\mathrm{span}(v_1, v_2, \ldots, v_N) \subset \mathrm{span}(v_1, v_2, \ldots, v_N, v_{N+1})$$

is clear. To prove equality does not hold, we argue by contradiction. Assume that $\mathrm{span}(v_1, v_2, \ldots, v_N) = \mathrm{span}(v_1, v_2, \ldots, v_N, v_{N+1})$ for some N. Then we could write

$$v_{N+1} = c_1 v_1 + c_2 v_2 + \ldots + c_N v_N$$

for some $c_1, \ldots, c_N$. But this implies that

$$c_1 v_1 + c_2 v_2 + \ldots + c_N v_N - v_{N+1} = 0$$

which contradicts the independence hypothesis. □

§3.5 Calculations Involving Linear Dependence

In linear algebra, we are often confronted with the following question:

> Let $v_1, v_2, \ldots, v_k$ be vectors in $\mathbb{R}^n$. Are $v_1, v_2, \ldots, v_k$ linearly dependent?

In other words, are there scalars $c_1, \ldots, c_k$, not all zero, such that $c_1 v_1 + c_2 v_2 + \ldots + c_k v_k = 0$? First of all, we write out the components of all these vectors

$$v_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} \quad \ldots \quad v_k = \begin{pmatrix} a_{1k} \\ \vdots \\ a_{nk} \end{pmatrix}.$$

Our equation then may be written in matrix form:

$$c_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} + \ldots + c_k \begin{pmatrix} a_{1k} \\ \vdots \\ a_{nk} \end{pmatrix} = \begin{pmatrix} a_{11} & \ldots & a_{1k} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nk} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Notice that $(c_1, \ldots, c_k)$ is a solution if and only if it is in the kernel of the matrix whose columns are equal to $v_1, \ldots, v_k$. Unwinding the matrix notation, we obtain

$$\begin{aligned} a_{11}c_1 + a_{12}c_2 + \ldots + a_{1k}c_k &= 0 \\ &\vdots \\ a_{n1}c_1 + a_{n2}c_2 + \ldots + a_{nk}c_k &= 0. \end{aligned}$$

To repeat, the vectors $v_1, \ldots, v_k$ are dependent if and only if these equations have a nonzero solution $(c_1, c_2, \ldots, c_k)$.

Example: Let $v_1 = (1, 2, 3), v_2 = (2, 1, 2), v_3 = (1, 1, -1) \in \mathbb{R}^3$. Following the discussion above, we obtain the equations

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Writing these out, we obtain

$$c_1 + 2c_2 + c_3 = 0 \quad 2c_1 + c_2 + c_3 = 0 \quad 3c_1 + 2c_2 - c_3 = 0.$$

Solving, we find that $c_1 = c_2 = c_3 = 0$ is the only solution, so v_1, v_2, and v_3 are linearly independent.

Exercises

1)Let $v_1 = e_1, v_2 = e_1 + e_2, v_3 = e_1 + e_2 + e_3$, and $v_4 = e_1 + e_2 + e_3 + e_4$ be vectors in $\mathbb{R}^4$.

a)Show that $w = e_1 + 2e_2 + 3e_3 + e_4 \in \text{span}(v_1, v_2, v_3, v_4)$.

b)Show that $w = e_1 + 2e_2 + 3e_3 + e_4 \notin \text{span}(v_1, v_2, v_4)$.

c)Show that v_1, v_2, v_3, v_4 are linearly independent.

2)Let $\mathcal{P}$ be the vector space of polynomials in x. Let $p_1 = 1 - x,\ p_2 = 1 - x^2$, and $p_3 = 1 - x^3$.

a)Show that p_1, p_2, p_3 are linearly independent.

b)Show that $1 + x + x^2 \notin \text{span}(p_1, p_2, p_3)$.

c)Show that $\text{span}(p_1, p_2, p_3) = \text{span}(1 - x, 1 - 2x + x^2, 1 - 3x + 3x^2 - x^3)$.

3)Let $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ denote counterclockwise rotation through an angle θ. Recall that T_θ has matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Verify $T_{\theta_1} \circ T_{\theta_2} = T_{\theta_1 + \theta_2}$ by multiplying matrices.
Let $J : \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection across the y-axis. Compute a matrix for J and verify that $J \circ T_\theta = T_{-\theta} \circ J$.

4)Let $S : V_1 \to V_2$ and $T : V_2 \to V_3$ be linear transformations.

a)Show that $\ker(S) \subset \ker(T \circ S)$.

b)Show that $\text{im}(T \circ S) \subset \text{im}(T)$.

c)Show that $T \circ S = 0$ iff $\text{im}(S) \subset \ker(T)$.

5)Let $T : V \to W$ be an injective linear transformation and let $v_1, v_2, \ldots, v_k$ be linearly independent vectors in V. Show that $T(v_1), T(v_2), \ldots, T(v_k)$ are linearly independent as well.