# §1 Introduction to Vector Spaces

## §1.1 Essential Examples

Before we dive into the axioms and proofs, we give some examples of vector spaces. Keep these examples in mind as you read the definitions which follow.

Example 1: $\mathbb{R}^n$

We define $\mathbb{R}^n$ to be the set of ordered n-tuples of real numbers, i.e.

$$\mathbb{R}^n := \{(x_1, x_2 \dots x_n) : x_1, \dots, x_n \in \mathbb{R}\}.$$

The elements of $\mathbb{R}^n$ are called *vectors*. Given a vector $x = (x_1, \dots, x_n)$, the numbers $x_1, \dots, x_n$ are called the *components* of $x$.

You are already quite familiar with $\mathbb{R}^n$ for small values of $n$. For example, $\mathbb{R}^1 = \mathbb{R}$ may be interpretted as the number line and $\mathbb{R}^2$ as the Cartesian plane. Furthermore, $\mathbb{R}^3$ can be interpretted as three-dimensional space; the components $x_1, x_2$ and $x_3$ correspond to the $x, y$ and $z$ coordinates of space.

We are interested in the algebraic properties of these sets which can be abstracted away from a concrete geometric interpretation. Specifically, we shall focus on two operations:

*Addition* Given two vectors $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, we may add them componentwise to obtain a new vector $x + y \in \mathbb{R}^n$ :

$$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n).$$

*Scalar Multiplication* Given a vector $x = (x_1, \dots, x_n)$ and a real number $r \in \mathbb{R}$, we may multiply $x$ by $r$ componentwise to obtain a new vector $rx \in \mathbb{R}^n$ :

$$rx = (rx_1, \dots, rx_n).$$

The real number $r$ is called a *scalar*, hence the terminology scalar multiplication. (You might try to guess the historical origin of this terminology by thinking about scalar multiplication geometrically in the case $n = 2$.)

Example 2: Sequences

Let $\mathcal{S}$ denote the set of all sequences of real numbers

$$\mathcal{S} := \{(s_1, s_2, \dots, s_n, \dots)\}$$

We call the elements of $\mathcal{S}$ *vectors*. The set $\mathcal{S}$ does not have the simple geometrical interpretations of $\mathbb{R}^n$, but it has many of the same algebraic properties. Specifically, we have the following operations on $\mathcal{S}$.

*Addition* Given two vectors $s = (s_n), t = (t_n) \in \mathcal{S}$, we may add them term by term to obtain a new vector $s + t \in \mathcal{S}$ :

$$s + t = (s_1 + t_1, s_2 + t_2, \dots, s_n + t_n, \dots).$$

*Scalar Multiplication* Given a vector $s = (s_n) \in \mathcal{S}$ and a real number $r \in \mathbb{R}$, we may multiply $s$ by $r$ term by term to obtain a new vector $rs \in \mathcal{S}$

$$rs = (rs_1, \dots, rs_n, \dots).$$

## §1.2 Vector Space Axioms

We give axioms for the common algebraic properties of these examples.

**Definition** A $\mathbb{R}$-*vector space* is a set $V$ equipped with two operations, called *addition* and *scalar multiplication*. The addition operation is a function which associates to each pair of vectors $v, w \in V$ the sum $v + w \in V$. The scalar multiplication is a function associating to each $r \in \mathbb{R}$ and each $v \in V$ the product $rv \in V$. The addition operation satisfies the following axioms:

1. $v + w = w + v$ for any $v, w \in V$ (Commutative Law)
2. $(u + v) + w = u + (v + w)$ for any $u, v, w \in V$ (Associative Law)
3. There exists a vector $0 \in V$ with the property that $v + 0 = v$ for any $v \in V$ (Zero Element)
4. For any $v \in V$ there exists a vector $-v \in V$ such that $v + (-v) = 0$ (Additive Inverse)

Furthermore, scalar multiplication satisfies the following axioms:

1. $1v = v$ for any $v \in V$(Preservation of Scale)
2. $(rs)v = r(sv)$ for any $r, s \in \mathbb{R}, v \in V$(Associative Law)
3. $(r + s)v = rv + sv$ for any $r, s \in \mathbb{R}, v \in V$ (Distributive Law I)
4. $r(v + w) = rv + rw$ for any $r \in \mathbb{R}, v, w \in V$ (Distributive Law II)

More generally, if $F$ is any field then we can define the notion of an $F$-vector space by taking scalars from $F$. However, the term 'vector space' here means an $\mathbb{R}$-vector space unless we say otherwise.

Obviously, checking that all these properties hold for a given set is quite laborious. Consequently, we leave much of the proof of the following result as an exercise:

**Theorem** $\mathbb{R}^n$ is a vector space (with respect to the operations described above).

*proof* Properties of Addition: We verify that addition is commutative. Let $v = (v_1, \ldots, v_n), w = (w_1, \ldots, w_n)$ be two vectors from $\mathbb{R}^n$; we have that

$$v + w = (v_1 + w_1, \ldots, v_n + w_n) = (w_1 + v_1, \ldots, w_n + v_n) = w + v.$$

We leave associativity to the reader. As for property three, we set $0 = (0, 0, \ldots, 0)$ i.e. the vector with components all equal to zero. For any $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ we have

$$v + 0 = (v_1, \ldots, v_n) + (0, \ldots, 0) = (v_1 + 0, \ldots, v_n + 0) = (v_1, \ldots, v_n) = v.$$

To prove property four, we set $-v = (-v_1, \ldots, -v_n)$. We check easily that

$$v+(-v) = (v_1, \ldots, v_n)+(-v_1, \ldots, -v_n) = (v_1-v_1, \ldots, v_n-v_n) = (0, \ldots, 0) = 0.$$

Properties of Scalar Multiplication: We verify properties two and three, leaving the first and the last to the reader. Let $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ and $r, s \in \mathbb{R}$. For the associativity, we compute

$$(rs)v = (rsv_1, \dots, rsv_n) = r(sv_1, \dots, sv_n) = r(sv).$$

As for the first distributive law,

$$(r+s)v = ((r+s)v_1, \dots, (r+s)v_n) = (rv_1, \dots, rv_n) + (sv_1, \dots, sv_n) = rv + sv.$$

This completes the proof. □

We should emphasize that for a set $V$ to be a vector space, the set $V$ must be closed under the two operations. In other words, for any $v, w \in V$ $v + w \in V$, and for any $r \in \mathbb{R}$ $rv \in V$ as well. For example, let $S \subset \mathbb{R}^2$ be the set of vectors in the first quadrant

$$S = \{(x, y) \in \mathbb{R}^2 : x, y > 0\}$$

and let addition and scalar multiplication be the same as for $\mathbb{R}^2$. Then $S$ is *not* a vector space, because it is not closed under scalar multiplication. Indeed, $(1, 1) \in S$ but $-(1, 1) = (-1, -1) \notin S$.

There is a systematic method for determining whether a subset of a vector space $V$ is also a vector space (with respect to the addition and scalar multiplication defined on $V$). This method will help us to prove that certain sets are vector spaces:

**Definition** Let $V$ be a vector space and let $S \subset V$ be a nonempty subset. Then we say $S$ is a *subspace* of $V$ if it satisfies the following properties:

1. *Closure under Addition:* For any $v, w \in S$, we have $v + w \in S$.
2. *Closure under Scalar Multiplcation:* For any $v \in S$ and any scalar $r$, we have $rv \in S$.

**Theorem** Let $V$ be a vector space and let $S \subset V$ be a subspace. Then $S$ itself is a vector space, with respect to the addition and multiplication defined on $V$.

*proof* Since $S$ is closed under addition and multiplication, the addition and multiplication from $V$ restrict to well-defined operations on $S$. For any vectors $u, v, w \in V$ and scalars $r, s$ we have the following identities:

$$v + w = w + v \quad (u + v) + w = u + (v + w)$$

$$1v = v \quad (rs)v = r(sv) \quad (r + s)v = rv + sv \quad r(v + w) = rv + rw.$$

Because $S \subset V$, these identities hold for all elements of $S$ as well.

We are left with two properties of addition to check. Namely, we have to show that $0 \in S$ and that if $v \in S$ then $-v \in S$. Note that multiplying any vector by the scalar zero gives the zero vector, i.e. $(0)v = 0$ for any $v$. Let $v \in S$; since $S$ is closed under scalar multiplication we have $0 = (0)v \in S$, i.e.

$S$ contains the zero vector. By the same reasoning, we obtain that $(-1)v \in S$. But $(-1)v = -v$ for any $v$, so $-v \in S$. $\square$

We should point out that every vector space $V$ contains itself as a subspace. Furthermore, the set $\{0\}$ containing just the zero vector is also a subspace, called the *trivial subspace.*

**§1.3 Further Examples**

Following is a list of examples of vector spaces, including a description of the operations addition and scalar multiplication.

> Let $\mathcal{S}$ be the set of sequences of real numbers with the termwise addition and scalar multiplication discussed above. The zero vector of $\mathcal{S}$ is the constant sequence with all terms zero. Then $\mathcal{S}$ is a vector space.
>
> Let $\mathcal{S}_c \subset \mathcal{S}$ be the subset consisting of convergent sequences. The termwise sum of two convergent series is convergent and any multiple of a convergent series is convergent, so $\mathcal{S}_c$ is closed under addition and scalar multiplication. Thus $\mathcal{S}_c$ is a subspace of $\mathcal{S}$, and so is a vector space.
>
> Let $\mathcal{P}$ be the set of all polynomials in a single variable $t$
>
> $$\mathcal{P} = \{a_0 + a_1 t + \ldots a_d t^d\}.$$
>
> Addition is defined in the normal way, coefficient by coefficient:
>
> $$\begin{aligned}(a_0 + a_1 t + \ldots a_d t^d) \quad &+ \quad (b_0 + b_1 t + \ldots b_d t^d) \\ &= \quad (a_0 + b_0) + (a_1 + b_1)t + \ldots + (a_d + b_d)t^d.\end{aligned}$$
>
> (If a monomial $t^k$ does not appear in a polynomial, then the corresponding coefficient is zero!) Scalar multiplication is defined by the rule
>
> $$r(a_0 + a_1 t + \ldots a_d t^d) = ra_0 + ra_1 t + \ldots ra_d t^d.$$
>
> Of course, the zero vector is the polynomial which is identically zero.

In doing analysis, you will come across a huge number of different vector spaces. These examples only scratch the surface.

**Exercises**

1)Using the axioms, verify the following properties of vector spaces

a)The zero element $0 \in V$ is unique.

b)For any $v \in V$, $(-1)v = -v$ and $(0)v = 0$.

2)Determine whether the following subsets of $\mathbb{R}^2$ are vector spaces. Please justify your answers.

a)$S = \{(x, y) \in \mathbb{R}^2 : x + y = 1\}$

b)$S = \{(x, y) \in \mathbb{R}^2 : x + y = 0\}$

3)Show that every subspace $V \subset \mathbb{R}^2$ is one of the following:

1. $V = \{0\}$
2. $V = \{(x, y) : (x, y) = (ta, tb) \text{ for some } t \in \mathbb{R}\}$ where $a, b \in \mathbb{R}$ and are not both zero (i.e. $V$ is a line through the origin)
3. $V = \mathbb{R}^2$

Hint: To prove this it suffices to show that if $V$ is a subspace and $V \neq \{0\}$ or $\mathbb{R}^2$, then the second case must hold.

4)Let $P \subset \mathcal{S}$ denote the set of sequences of positive real numbers. Is $P$ a vector space (with respect to the operations of termwise addition and scalar multiplication)? Justify your answer.

5)Let $\mathcal{S}_0 \subset \mathcal{S}$ denote the set of sequences that converge to 0. Is $\mathcal{S}_0$ a subspace of $\mathcal{S}$? Prove your answer.

6)Let $\mathcal{P}_d$ denote the set of all polynomials in $t$ with degree less than or equal to $d$, i.e.

$$\mathcal{P}_d = \{a_d t^d + a_{d-1} t^{d-1} + \ldots + a_0 \text{ where } a_0, \ldots, a_d \in \mathbb{R}\}.$$

Show that $\mathcal{P}_d$ is vector space (with respect to the addition and scalar multiplication defined above for polynomials.)