Problem 1-33: Consider two open balls $B(a,r)$ and $B(b,s)$ in the case where $\|a-b\| = r+s$. Then the two balls are tangent at x. What is x equal to in terms of a, b, r, and s?

The point x is on the line segment $[a,b]$, thus $x = (1-t)a + tb$ for some $t \in [0,1]$. The conditions $\|a-x\| = r$ and $\|b-x\| = s = \|a-b\| - r$ translate into

$$\|t(a-b)\| = r \quad \|(1-t)(b-a)\| = \|a-b\| - r.$$

These are equivalent to $t = r/\|a-b\|$ so

$$x = (1 - r/\|a-b\|)a + rb/\|a-b\| = a + r(b-a)/\|a-b\|.$$

Problem 1-35: Prove that for $n \geq 2$ the sphere $S(a,r)$ is quite "round", in the sense that there do not exist three distinct points in $S(a,r)$ which are collinear.

Applying the translation

$$x \mapsto x - a$$

and the rescaling

$$x \mapsto x/r$$

takes the sphere $S(a,r)$ to $S(0,1)$, and takes triples of collinear points to triples of collinear points. We therefore restrict to the case $a = 0$ and $r = 1$.

Suppose we have distinct collinear $x, y, z \in \mathbb{R}^n, n \geq 2$ with $x, y, z \in S(0,1)$, i.e.,

$$z = tx + (1-t)y, \quad \|x\| = \|y\| = \|z\| = 1.$$

We have

$$\begin{aligned}
1 = \|z\|^2 &= (tx + (1-t)y) \cdot (tx + (1-t)y) \\
&= t^2 x \cdot x + 2t(1-t)x \cdot y + (1-t)^2 y \cdot y \\
&= t^2 + 2t(1-t)x \cdot y + (1-t)^2 \\
&\leq t^2 + |2t(1-t)x \cdot y| + (1-t)^2 \\
&\leq t^2 + |2t(1-t)|\|x\|\|y\| + (1-t)^2 \text{ Schwartz ineq.} \\
&\leq t^2 + |2t(1-t)| + (1-t)^2.
\end{aligned}$$

If $t < 0$ or $t > 1$ then the last term is

$$t^2 - 2t(1-t) + (1-t)^2 = 4t^2 - 4t + 1 = (2t-1)^2,$$

which is greater than 1, a contradiction. If $0 \le t \le 1$ then the last term is

$$t^2 + 2t(1-t) + (1-t)^2 = 1.$$

There is no contradiction provided the Schwartz inequality is an equality. This can only happen when $x = cy$ for some scalar $c > 0$. However, since x and y have the same norm $c = 1$, which violates the hypothesis that x and y are distinct.

Problem 1-36:

Assume $n \ge 2$. Prove that the two spheres $S(a, r)$ and $S(b, s)$ have a nonempty intersection if and only if $|r - s| \le \|a - b\| \le r + s$. Explain why the intersection is like a sphere of radius R in $\mathbb{R}^{n-1}$, where

$$R^2 = \frac{r^2 + s^2}{2} - \frac{\|a-b\|^2}{4} - \frac{(r^2 - s^2)^2}{4\|a-b\|^2}.$$

First, let's consider the 'only if' part. The condition $\|a - b\| \le r + s$ follows from Problem 1-31 in the last homework set. If $x \in S(a, r), S(b, s)$ then the triangle inequality gives

$$s = \|x - b\| \le \|x - a\| + \|a - b\| = r + \|a - b\|$$

and

$$r = \|x - a\| \le \|x - b\| + \|a - b\| = s + \|a - b\|,$$

i.e., $|r - s| \le \|a - b\|$.

For the 'if' part, we need to produce points in the intersection of the spheres when the conditions are satisfied. We claim that every point $x \in \mathbb{R}^n$ has a unique expression

$$x = (1-t)a + tb + u, \quad (a - b) \perp u,$$

provided $a \ne b$. Equivalently, there is a unique value $t \in \mathbb{R}$ so that

$$(x - (1-t)a - tb) \cdot (a - b) = 0.$$

Indeed, this is the case precisely when

$$t = (x - a) \cdot (b - a)/\|b - a\|^2.$$

In order that $x \in S(a,r), S(b,s)$, we must have

$$r^2 = (x-a)\cdot(x-a) = t^2\|b-a\|^2 + u\cdot u$$

and

$$s^2 = (x-b)\cdot(x-b) = (1-t)^2\|b-a\|^2 + u\cdot u.$$

Subtracting these yields

$$r^2 - s^2 = \|b-a\|^2(2t-1)$$

hence

$$t = 1/2(r^2 - s^2 + \|b-a\|^2)/\|b-a\|^2.$$

If we set $R = \|u\|$ then

$$\begin{aligned} R^2 &= r^2 - t^2\|b-a\|^2 \\ &= r^2 - 1/4(r^2 - s^2 + \|a-b\|^2)^2/\|a-b\|^2 \\ &= r^2 - 1/4(r^2-s^2)^2/\|a-b\|^2 - (r^2-s^2)/2 - 1/4\|a-b\|^2 \\ &= (r^2+s^2)/2 - \|a-b\|^2/4 - \frac{(r^2-s^2)^2}{4\|a-b\|^2}. \end{aligned}$$

Thus the points of intersection correspond to vectors u with $\|u\| = R$ and $u\cdot(a-b) = 0$. (This is like a sphere of radius R in the $(n-1)$-dimensional space of vectors orthogonal to $(a-b)$.) These vectors exist provided $R \geq 0$, i.e., $-r \leq t\|a-b\| \leq r$. This is equivalent to

$$-r \leq 1/2(r^2 - s^2 + \|b-a\|^2)/\|b-a\|^2 \leq r$$

or

$$-2r - \|a-b\| \leq (r^2-s^2)/\|a-b\| \leq 2r - \|a-b\|.$$

Our assumptions are

$$-\|a-b\| \leq r - s \leq \|a-b\| \quad \|a+b\| \leq r+s$$

which imply

$$(r^2-s^2)/\|a-b\| \leq r+s \leq 2r - \|a-b\|$$

and

$$(r^2-s^2)/\|a-b\| \geq -(r+s) \geq -2r - \|a-b\|.$$

Thus we have the required inequalities.

Problem 1-37: Assume $n \geq 2$. Given two intersecting spheres $S(a, r)$ and $S(b, s)$, say that the *angle* between them is the angle formed at a common point x by the vectors $a - x$ and $b - x$. Prove that this angle θ is independent of x and satisfies

$$\cos\theta = \frac{r^2 + s^2 - \|a - b\|^2}{2rs}.$$

Consider the triangle with vertices x, a, b. The law of cosines

$$\|a - b\|^2 = \|a - x\|^2 + \|b - x\|^2 - 2\|a - x\|\|b - x\| \cos\theta$$

can be expressed

$$\|a - b\|^2 = r^2 + s^2 - 2rs \cos\theta.$$

Solving for θ gives the desired formula, which does not depend on x.