Problem 1-23: Consider a triangle with vertices $x, y, z \in \mathbb{R}^n$ and sides of lengths

$$a = \|y - z\| \quad b = \|x - z\| \quad c = \|x - y\|.$$

Prove that the angle bisectors of the three angles of the triangle are concurrent, intersecting at the point

$$p := \frac{ax + by + cz}{a + b + c}.$$

It suffices to show that

$$\angle yxp = \angle pxz \quad \angle yzp = \angle pzx \quad \angle xyp = \angle pyz.$$

By symmetry it suffices to prove the first equality. Let $\theta = \angle yxp$ and $\phi = \angle pxz$; these are equal if $\cos\theta = \cos\phi$.

By the angle formula

$$\cos\theta = \frac{(y - x) \cdot (p - x)}{\|y - x\| \|p - x\|}$$

and

$$\cos\phi = \frac{(p - x) \cdot (z - x)}{\|p - x\| \|z - x\|}$$

so we are reduced to verifying

$$\frac{(y - x) \cdot (p - x)}{\|y - x\| \|p - x\|} = \frac{(p - x) \cdot (z - x)}{\|p - x\| \|z - x\|}.$$

This is equivalent to

$$(y - x) \cdot (p - x)b = (p - x) \cdot (z - x)c.$$

The lefthand side is

$$(y-x)\cdot(-(b+c)x+by+cz)b/(a+b+c) = [b(y-x)\cdot(y-x)+c(z-x)\cdot(y-x)]b/(a+b+c);$$

the righthand side is

$$[b(y - x) \cdot (z - x) + c(z - x) \cdot (z - x)]c/(a + b + c).$$

The desired equality is thus equivalent to

$$b^2(y-x)\cdot(y-x)+bc(z-x)\cdot(y-x) = bc(y-x)\cdot(z-x)+c^2(z-x)\cdot(z-x). \quad (1)$$

Since $(y-x)\cdot(y-x) = \|y-x\|^2 = c^2$ and $(z-x)\cdot(z-x) = \|z-x\|^2 = b^2$, equation (1) holds true.

Problem 1-24: Consider a triangle in $\mathbb{R}^2$ with vertices $x, y, z$. This triangle has a unique circumcircle, the circle which passes through all three vertices. Its center $q$ is called the *circumcenter* of the triangle. Prove that the point $p = x+y+z-2q$ is the *orthocenter*, the common intersection of the altitudes.

The altitudes are the perpendiculars drawn from one of the vertices to the line determined by the other two. To show that $p$ is on the altitude from $x$ to the line $\ell(y,z)$, it suffices to show that the line $\ell(p,x)$ is perpendicular to $\ell(y,z)$ (or $p = x$). Algebrically, this entails verifying

$$(p-x)\cdot(y-z) = 0.$$

The other cases follow by symmetry.

Since $q$ is the circumcenter, we have

$$\|q-x\| = \|q-y\| = \|q-z\|.$$

It follows that

$$(q-y)\cdot(q-y) = (q-z)\cdot(q-z)$$

which means

$$-2q\cdot y + y\cdot y = -2q\cdot z + z\cdot z$$

and thus

$$(p-x-y-z)\cdot y + y\cdot y = (p-x-y-z)\cdot z + z\cdot z.$$

Expanding and cancelling gives

$$p\cdot y - x\cdot y = p\cdot z - x\cdot z$$

which is equivalent to

$$(p-x)\cdot(y-z) = 0.$$

Problem 1-29 Show that the area of a general convex quadrilateral in $\mathbb{R}^2$ cannot be expressed as a function of the lengths of the four sides.

Take one quadrilateral to be the square with sides of length one, which has area one. For the other, take a rhombus with sides of length one and interior angles $\theta, \pi - \theta, \theta, \pi - \theta$. (This is a square when $\theta = \pi/2$.) The diagonals of the rhombus divide into four right triangles, with hypotenuse one and angles $\theta/2, \pi/2 - \theta/2, \pi/2$. Each triangle has area

$$\frac{1}{2}\cos(\theta/2)\sin(\theta/2)$$

and so the rhombus has area

$$2\cos(\theta/2)\sin(\theta/2) = \sin\theta.$$

The last equality is the double-angle identity of trigonometry. For example, when $\theta = \pi/4$ the area is $\sqrt{2}/2$.

Problem 1-31 Prove that the two closed balls $\bar{B}(a,r)$ and $\bar{B}(b,s)$ have a nonempty intersection if and only if $\|a-b\| \le r+s$.

First, suppose that the two balls have a point $p$ in common, so that $\|a-p\| \le r$ and $\|b-p\| \le s$. Then by the triangle inequality

$$\|a-b\| \le \|a-p\| + \|p-b\| \le r+s.$$

Conversely, suppose that $\|a-b\| \le r+s$. Since we have

$$\|a-b\| - s \le r$$

we can choose $t$ so that

$$\|a-b\| - s \le t \le r.$$

Set

$$p = a + t(b-a)/\|b-a\|$$

so that

$$\|a-p\| = \|t(b-a)/\|b-a\|\| = t \le r$$

and

$$\|b-p\| = \|(b-a) - t(b-a)/\|b-a\|\| = \|b-a\| - t \le s,$$

i.e., $p \in \bar{B}(a,r)$ and $p \in \bar{B}(b,s)$.

**Extra problem:** Show that any plane $P$ is convex, i.e., for any distinct points $v$ and $w$ on $P$, the segment $[v,w]$ lies completely in $P$.

Each plane $P$ can be expressed as the locus of points

$$\{t_1x + t_2y + t_3z : t_1 + t_2 + t_3 = 1\},$$

where $x, y, z \in \mathbb{R}^n$ are non-collinear. Given

$$v = t_1x + t_2y + t_3z \quad w = u_1x + u_2y + u_3z \in P,$$

we want to show that $\tau v + (1 - \tau)w \in P$ for each $\tau \in [0, 1]$. Expanding

$$\tau v + (1 - \tau)w = (\tau t_1 + (1 - \tau)u_1)x + (\tau t_2 + (1 - \tau)u_2)y + (\tau t_3 + (1 - \tau)u_3)z$$

and the coefficients sum to one:

$$\begin{aligned} \tau t_1 + (1 - \tau)u_1 + \tau t_2 + (1 - \tau)u_2 + \tau t_3 + (1 - \tau)u_3 &= \\ \tau(t_1 + t_2 + t_3) + (1 - \tau)(u_1 + u_2 + u_3) &= \\ \tau + (1 - \tau) &= 1. \end{aligned}$$