Problem 1-15: Consider two diagonals of faces of a cube which intersect at a vertex. Compute the angle between them.

We refer to the figure in Problem 1-16; the vertices of the cube are

$$\{(\pm 1, \pm 1, \pm 1)\}.$$

Take the vertex to be $v = (1, -1, -1)$, and the diagonals to be the line segments joining v to $w = (-1, -1, 1)$ and $x = (1, 1, 1)$ respectively. The angle formula on page 18 gives

$$\cos\theta = \frac{(w-v)\cdot(x-v)}{\|w-v\|\|x-v\|} == \frac{(-2,0,2)\cdot(0,2,2)}{\sqrt{8}\sqrt{8}} = 1/2$$

hence $\theta = 60°$.

Problem 1-17: Given the tetrahedron with vertices $(1,0,0,0)$,$(0,1,0,0)$,$(0,0,1,0)$, and $(0,0,0,1)$ and centroid $(1/4,1/4,1/4,1/4)$, consider the segments joining the centroid with two of the vertices. Calculate the angle they form.

Take $v = (1/4,1/4,1/4,1/4)$, $w = (1,0,0,0)$ and $x = (0,1,0,0)$ and applying the formula above we find

$$\cos\theta = \frac{(3/4,-1/4,-1/4,-1/4)\cdot(-1/4,3/4,-1/4,-1/4)}{\|(3/4,-1/4,-1/4,-1/4)\|\|(-1/4,3/4,-1/4,-1/4)\|} = \frac{-4/16}{12/16} = -1/3$$

so that $\theta \simeq 109.47°$.

Problem 1-20: Use the triangle inequality to prove that for any points $x, y, z \in \mathbb{R}^n$

$$d(x,y) \le d(x,z) + d(z,y).$$

And prove that equality holds $\iff$ z belongs to the line segment $[x,y]$.

The triangle inequality says

$$\|v+w\| \le \|v\| + \|w\|; \tag{1}$$

we saw equality holds only when one of the vectors is a nonnegative scalar of the other. Taking $v = x - z$ and $w = z - y$ we find

$$\|x-y\| \le \|x-z\| + \|z-y\|,$$

and the definition of distance gives the desired inequality.

Suppose first that equality holds. We have either $(x-z)=t(z-y), t\geq 0$ or $(z-y)=s(x-z), s\geq 0$; we do the first case, leaving the second to the reader. Regrouping gives

$$z(t+1)=x+ty$$

and

$$z=\frac{1}{t+1}x+\frac{t}{t+1}y.$$

Note that

$$\frac{1}{t+1}+\frac{t}{t+1}=1$$

and both terms are between 0 and 1 when $t\geq 0$. Hence we have shown that $z\in[x,y]$.

Now suppose that $z\in[x,y]$, so that

$$z=\tau x+y(1-\tau),\quad 0\leq\tau\leq 1.$$

It follows that

$$(1-\tau)(z-y)=\tau(x-z).$$

If $\tau=0$ then $z=y$ and the equality of distances is immediate. Otherwise, we can write

$$(x-z)=\frac{1-\tau}{\tau}(z-y);$$

moreover, since $0<\tau\leq 1$ we have

$$\frac{1-\tau}{\tau}\geq 0$$

and equality holds in the triangle inequality.

Problem 1-21: Prove that for any $x,y\in\mathbb{R}^n$,

$$|\|x\|-\|y\||\leq\|x-y\|.$$

Also prove that for any $x,y,z\in\mathbb{R}^n$,

$$|d(x,y)-d(x,z)|\leq d(y,z).$$

Unwinding the definition of the absolute value, the desired inequality is equivalent to

$$-\|x-y\|\leq\|x\|-\|y\|\leq\|x-y\|.$$

The first of these is equivalent to

$$\|y\| \leq \|x\| + \|x - y\| = \|x\| + \|y - x\|$$

which is just the triangle inequality (1) for the vectors $v = x$ and $w = y - x$. The second inequality is equivalent to

$$\|x\| \leq \|x - y\| + \|y\|$$

which is the triangle inequality for $v = x - y$ and $w = y$.

For the last assertion, we apply the newly proven inequality to $x = x' - y', y = z' - x'$.

Problem 1-22: Prove the **parallelogram law**: The sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its edges.

The solution to this problem refers to the diagram on page 24; the sides of this parallelogram have lengths $\|x\|$ and $\|y\|$. The diagonal pointing 'northeast' is the vector $x + y$; the diagonal pointing 'northwest' from x to y is the vector $y - x$. The statement we want to prove is encapsulated in the equation

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

To prove this, we expand all the norms out in terms of inner products:

$$\begin{aligned}
\|x + y\|^2 + \|x - y\|^2 &= (x + y) \cdot (x + y) + (x - y) \cdot (x - y) \\
&= x \cdot x + 2x \cdot y + y \cdot y + x \cdot x - 2x \cdot y + y \cdot y \\
&= 2x \cdot x + 2y \cdot y \\
&= 2\|x\| + 2\|y\|.
\end{aligned}$$