Extra problem: Show there do not exist nonzero vectors $x, y, z \in \mathbb{R}^2$ which are pairwise orthogonal, i.e., x is orthogonal to y, y is orthogonal to z, and z is orthognal to x.

Let $x = (x_1, x_2), y = (y_1, y_2)$, and $z = (z_1, z_2)$. The orthogonality assumption gives

$$x \cdot y = y \cdot z = z \cdot x = 0,$$

which translates into the formulas

$$x_1y_1 + x_2y_2 = y_1z_1 + y_2z_2 = z_1x_1 + z_2x_2 = 0.$$

Since z is nonzero, one of its coordinates, say z_1, is nonzero. The formulas above imply that

$$0 = x_2(y_1z_1 + y_2z_2) - y_2(z_1x_1 + z_2x_2) = z_1(x_2y_1 - x_1y_2)$$

so we find that $x_2y_1 = x_1y_2$. Since y is nonzero, either $y_1 \neq 0$ or $y_2 \neq 0$. If $y_1 \neq 0$ then we have

$$x_1 = -x_2y_2/y_1 \qquad x_2 = x_1y_2/y_1 \ (\dagger)$$

and combining these yields

$$x_1 = -x_1y_2^2/y_1^2.$$

In other words,

$$x_1(1 + y_2^2/y_1^2) = 0.$$

Since $(y_2/y_1)^2 \geq 0$, we have $1 + (y_2/y_1)^2 \neq 0$ and $x_1 = 0$. Equation ($\dagger$) above then implies that $x_2 = 0$. We leave the case $y_2 \neq 0$ to the reader.

Problem 1-10: Show that if $x \in \mathbb{R}^n$ is orthogonal to every vector then $x = 0$.

If $x \in \mathbb{R}^n$ is orthogonal to every vector in $\mathbb{R}^n$ then $x \cdot x = 0$. This means that $\|x\| = 0$, but the only vector of norm zero is $x = 0$.

Problem 1-14: Let $n \geq 2$ and let M be the subset of $\mathbb{R}^n$ consisting of all points of the form $x = (x_1, x_2, 0, \ldots, 0)$. (In other words, M is the x_1, x_2 plane.) Let $y \in \mathbb{R}^n$.

a. Find the unique $x \in M$ such that $y - x$ is orthogonal to all points in M.

b. Find the unique $x \in M$ which is closest to y.

Write $y = (y_1, y_2, \ldots, y_n)$ and take $x = (y_1, y_2, 0, \ldots, 0)$.
For part a, for each $z = (z_1, z_2, 0, \ldots, 0) \in M$ we have

$$(y - x) \cdot z = (0, 0, y_3, \ldots, y_n) \cdot (z_1, z_2, 0 \ldots, 0) = 0,$$

i.e., $y - x$ is orthogonal to z. For completeness, we'll prove uniqueness: Suppose $w = (w_1, w_2, 0, \ldots, 0) \in M$ is such that $y - w$ is orthogonal to each $z \in M$. Then we have

$$(y - w) \cdot z = (y_1 - w_1)z_1 + (y_2 - w_2)z_2 = 0$$

for every z_1, z_2. This can only happen when $y_1 - w_1 = y_2 - w_2 = 0$, so $w = x$.

For part b, we will verify that $\|y - x\| \leq \|y - w\|$ for each $w \in M$. It suffices to show that $\|y - x\|^2 \leq \|y - w\|^2$ for each w. However, we have

$$\begin{aligned} \|y - w\|^2 &= (y_1 - w_1)^2 + (y_2 - w_2)^2 + y_3^2 + \ldots + y_n^2 \\ &\geq y_3^2 + \ldots + y_n^2 \\ &= \|y - x\|^2, \end{aligned}$$

with equality if and only if $y_1 = w_1$ and $y_2 = w_2$, i.e., if and only if $w = x$.