Problem 1-4: Consider a triangle with vertices $x, y, z \in \mathbb{R}^n$. Let L_1 denote the line determined by x and the midpoint $(y+z)/2$, L_2 the line determined by y and $(x+z)/2$, and L_3 the line determined by z and $(x+y)/2$. Show that these lines all meet in the centroid.

The centroid is the point $p = (x+y+z)/3$. Recall that

$$\begin{aligned} L_1 &= \{(1-t_1)x + t_1(y+z)/2, t_1 \in \mathbb{R}\} \\ L_2 &= \{(1-t_2)y + t_2(x+z)/2, t_2 \in \mathbb{R}\} \\ L_3 &= \{(1-t_3)z + t_3(x+y)/2, t_3 \in \mathbb{R}\}. \end{aligned}$$

We can write

$$\frac{x+y+z}{3} = (1-2/3)x + 2/3\frac{y+z}{2} = (1-2/3)y + 2/3\frac{x+z}{2} + (1-2/3)z + 2/3\frac{x+y}{2}$$

so that $p \in L_1, L_2, L_3$ with $t_1 = t_2 = t_3 = 2/3$.

Problem 1-6: Prove that the four points w, x, y, z in $\mathbb{R}^n$ are *coplanar* if and only if there exist real numbers t_1, t_2, t_3, t_4, not all zero, such that

$$\begin{aligned} t_1 + t_2 + t_3 + t_4 &= 0 \\ t_1 w + t_2 x + t_3 y + t_4 z &= 0. \end{aligned}$$

Suppose that w, x, y, z are coplanar, i.e., there exist non-collinear points p, q, r so that w, x, y, z are all in the plane determined by p, q, and r. If three of w, x, y, z are collinear (say, w, x, y) then Problem 1.1 implies that there exist t_1, t_2, t_3, not all zero, with $t_1 + t_2 + t_3 = 0$ and $t_1 w + t_2 x + t_3 y = 0$. Taking $t_4 = 0$, we obtain the desired real numbers. So we may assume that no three of w, x, y, z are collinear. The rest of our argument will be much simpler once we know that the plane determined by p, q, and r coincides with the plane determined by w, x, and y. This might seem obvious, especially if one takes for granted that there is a unique plane through any three non-collinear points. However, we will state this here formally so that it can be proven in the future:

Lemma: There passes a unique plane through any three non-collinear points.

Allowing this, the fact that z lies on the plane determined by w, x, and y means we can write

$$z = s_1 w + s_2 x + s_3 y, \quad s_1 + s_2 + s_3 = 1.$$

We may therefore take $t_1 = s_1, t_2 = s_2, t_3 = s_3, t_4 = -1$.

Now suppose we have t_1, t_2, t_3, t_4 with the prescribed properties. One of these is nonzero, say, t_1. Then we can write

$$w = -t_2/t_1 x - t_3/t_1 y - t_4/t_1 z$$

with

$$(-t_2/t_1) + (-t_3/t_1) + (-t_4/t_1) = t_1/t_1 = 1,$$

so that w lies in the plane determined by x, y, z.

Problem 1-7: Consider the *quadrilateral* determined by the four distinct points w, x, y, z, with sides equal to the segments $[w, x], [x, y], [y, z], [z, w]$. Prove that the four midpoints of the sides of a quadrilateral, taken in order, form the vertices of a parallelogram.

Label the midpoints p_1, p_2, p_3, and p_4, i.e.,

$$p_1 = (w + x)/2,\ p_2 = (x + y)/2,\ p_3 = (y + z)/2,\ p_4 = (z + w)/2.$$

What does it mean to say that these determine a parallelogram? Each pair of opposite sides should have the same magnitutde and should point in the same direction. For the sides $[p_1, p_2]$ and $[p_3, p_4]$, the corresponding vector equation is

$$p_2 - p_1 = p_3 - p_4;$$

for the sides $[p_2, p_3]$ and $[p_4, p_1]$ the equation is

$$p_3 - p_2 = p_4 - p_1.$$

These are both equivalent to the relation

$$p_1 - p_2 + p_3 - p_4 = 0; \tag{1}$$

this also suffices to prove that p_1, p_2, p_3, p_4 are coplanar by problem 1.6.

We verify (1) by computing

$$p_1 - p_2 + p_3 - p_4 = (w + x)/2 - (x + y)/2 + (y + z)/2 - (z + w)/2 = 0.$$

Problem 1-9:

Give a careful proof that any three points in $\mathbb{R}^1$ are collinear and also that any four points in $\mathbb{R}^2$ are coplanar.

Consider the line in $\mathbb{R}^1$ determined by the points 0 and 1. We assert that *every* point in $\mathbb{R}^1$ lies on this line; in particular, any three points are collinear. Given $x \in \mathbb{R}^1$, we can write

$$x = (1-t)0 + t1$$

for some $t \in \mathbb{R}$; just take $t = x$!

Consider the plane in $\mathbb{R}^2$ determined by the points $(0,0)$, $(1,0)$, and $(0,1)$. Every point $\mathbb{R}^2$ lies on this plane. Given $x = (x_1, x_2)$, we need to find $t_1, t_2, t_3 \in \mathbb{R}$ with $t_1 + t_2 + t_3 = 1$ and

$$(x_1, x_2) = t_1(0,0) + t_2(1,0) + t_3(0,1).$$

We can take $t_2 = x_1, t_3 = x_2$, and $t_1 = 1 - x_1 - x_2$.