Problem 1-1: Prove that three points $x, y, z \in \mathbb{R}^n$ lie on a line if and only if there exist scalars t_1, t_2, t_3, not all zero, such that

$$
\begin{aligned}
t_1 + t_2 + t_3 &= 0 \qquad (1)\\
t_1 x + t_2 y + t_3 z &= 0. \qquad (2)
\end{aligned}
$$

First we prove the 'only if': Suppose that x, y, and z all lie on a line L, determined by the points $v, w \in \mathbb{R}^n$. By the definition of a line, there exist numbers $s_1, s_2, s_3 \in \mathbb{R}$ with

$$
\begin{aligned}
x &= (1-s_1)v + s_1 w\\
y &= (1-s_2)v + s_2 w\\
z &= (1-s_3)v + s_3 w.
\end{aligned}
$$

Setting $t_1 = s_2 - s_3, t_2 = s_3 - s_1$, and $t_3 = s_1 - s_2$, we have

$$t_1 + t_2 + t_3 = (s_2 - s_3) + (s_3 - s_1) + (s_1 - s_2) = 0$$

and

$$
\begin{aligned}
t_1 x + t_2 y + t_3 z &= [(s_2 - s_3)(1-s_1) + (s_3 - s_1)(1-s_2) + (s_1 - s_2)(1-s_3)]v\\
&\quad +[(s_2 - s_3)s_1 + (s_3 - s_1)s_2 + (s_1 - s_2)s_3]w\\
&= 0v + 0w.
\end{aligned}
$$

We are done except in the case $t_1 = t_2 = t_3 = 0$. Then we have $s_1 = s_2 = s_3$ and $x = y = z$. Here we can take the relation $x - y = 0$, i.e., $t_1 = 1, t_2 = -1, t_3 = 0$.

Now we prove the 'if': Without loss of generality, we may assume $t_1 \neq 0$. Relation 2 allows us to write

$$x = -t_2/t_1 y - t_3/t_1 z.$$

If we set $\tau = -t_3/t_1$ then relation 1 gives

$$1 - \tau = 1 + t_3/t_1 = (t_1 + t_3)/t_1 = -t_2/t_1$$

so that

$$x = (1-\tau)y + \tau z. \qquad \square$$

Problem 1-2: Prove that $x, y, 0$ are collinear if and only if x is a scalar multiple of y or y is a scalar multiple of x.

First, we prove the 'only if' part: Suppose $x, y, 0$ are collinear, so by the previous problem there exist $t_1, t_2, t_3 \in \mathbb{R}$, not all zero, with

$$\begin{aligned} t_1 + t_2 + t_3 &= 0 \\ t_1 x + t_2 y + t_3 0 = t_1 x + t_2 y &= 0. \end{aligned}$$

Now t_1 and t_2 cannot both be zero; indeed, if $t_1 = t_2 = 0$ then $t_3 = -t_1 - t_2 = 0$ as well. So without loss of generality we may assume $t_1 \neq 0$, which allows us to write

$$x = -t_2/t_1 y,$$

i.e., x is a scalar multiple of y.

Now for the 'if' part: Let's first assume x is a scalar multiple of y, i.e., $y = ax$ for some $a \in \mathbb{R}$. We have

$$ax - y + (1 - a)0 = 0$$

so this satisfies the conditions 1 and 2 for $t_1 = a, t_2 = -1, t_3 = 1 - a$. $\square$