FINITE GENERATION OF CANONICAL RINGS

Let $X \subset \mathbb{P}^N_{\mathbb{C}}$ be a smooth algebraic variety i.e. a sub-manifold defined by homogeneous polynomials $P_1, ..., P_t \in \mathbb{C}[z_0, ..., z_N]$. T_X denotes the tangent bundle to X and $\omega_X = \Lambda^{\mathrm{dimX}}(T_X^\vee)$ is the canonical bundle of X. $H^0(\omega_X^{\otimes m})$ denotes the space of global sections of the m-th tensor power of the canonical line bundle so that an element $s \in H^0(\omega_X^{\otimes n})$ can be written in local coordinates as $f(x_1, ..., x_n)(dx_1 \wedge ... \wedge dx_n)^{\otimes m}$ for some holomorphic function f.

The vector spaces $H^0(\omega_X^{\otimes m})$ play a fundamental role in understanding the geometry of X. If $\mathrm{dim}X = 1$, it is well known that $\mathrm{dim}H^0(\omega_X) = g$ is the geometric genus of X. If $\mathrm{dim}X = 2$, then the geometry of X was well understood (in terms of the groups $H^0(\omega_X^{\otimes m})$) by the Italian school of Algebraic Geometry around the beginning of the 20^{th} century. In particular, it is known that if X is not covered by curves of genus 0, then X is birational (i.e. isomorphic outside of a measure zero set) to a unique surface $\bar{X}$ for which $\omega_{\bar{X}}$ is semipositive (in the sense that $\deg(\omega_{\bar{X}}|_C) \geq 0$ for any curve $C \subset \bar{X}$). In this case, it then follows that the canonical ring $\oplus_{m \geq 0} H^0(\omega_X^{\otimes m})$ is finitely generated. In the 1980's, by celebrated results of Mori and others, these results were extended from $\mathrm{dim}X = 2$ to the case of $\mathrm{dim}X = 3$.

In this talk I will discuss joint work with Birkar, Cascini and M$^{\mathrm{c}}$Kernan towards understanding the geometry of algebraic varieties of arbitrary dimension. In particular I will discuss the following:

Theorem 0.1. *Let X be a smooth projective algebraic variety. Then the canonical ring*

$$R(X) = \bigoplus_{m \geq 0} H^0(\omega_X^{\otimes m})$$

is finitely generated.

Note that this Theorem was independently proven by Siu.