

Algebra Qualifying Exam

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1. Show that $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ has $p + 1$ subgroups of order p , when p is prime. Show that the group of automorphisms of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is isomorphic to S_3 by considering its action on the subgroups of order 2.
2. Let $f(x) = x^8 + 1$.
 - a. Let K be the splitting field of $f(x)$ over \mathbb{Q} , the field of rational numbers. Determine the Galois group of K/\mathbb{Q} .
 - b. How many subfields of K are of degree 4 over \mathbb{Q} ? How many of these are Galois over \mathbb{Q} ? Explain.
 - c. Let L be the splitting field of $f(x)$ over \mathbb{F}_{41} , the field of 41 elements. Determine the Galois group of L/\mathbb{F}_{41} .
3. Let p be a prime. Show that each group of order p^2 is abelian. Give an example of a non-abelian group of order p^3 for each prime p .
4. Let R be a PID and F its field of fractions. Suppose S is a ring with $R \subset S \subset F$.
 - a. Show that all elements $\alpha \in S$ can be written as a/b , where $a, b \in R$ and $1/b \in S$.
 - b. Show that S is a PID.
 - c. Show that if S is finitely generated as an R -module then $S = R$.
5. Let $V = \mathbb{R}^n$ and consider two sets of linearly independent vectors $\{v_1, v_2, v_3\}, \{w_1, w_2, w_3\} \subset V$.
 - a. Show that $v_1 \wedge v_2 \wedge v_3 = cw_1 \wedge w_2 \wedge w_3 \in \wedge^3 V$ for some $c \in \mathbb{R}$ if and only if $\text{span}(v_1, v_2, v_3) = \text{span}(w_1, w_2, w_3)$.
 - b. Does $\text{span}(v_1, v_2, v_3) = \text{span}(w_1, w_2, w_3)$ imply that $v_1 \otimes v_2 \otimes v_3 = cw_1 \otimes w_2 \otimes w_3 \in \mathcal{T}^3 V$, for some $c \in \mathbb{R}$?
6. Let $R = \mathbb{Q}[x, y]$ and $I = \langle x, y \rangle \subset R$ the ideal generated by x and y . Show that R/I is neither flat nor projective as an R -module. Do the same for I .