

# Algebra Qualifying Exam

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1. Let  $R$  be an integral domain with fraction field  $K$ .
  - a. Assume  $R$  is a unique factorization domain. Suppose that the monic polynomial

$$p(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0 \in R[x]$$

has a root  $\alpha \in K$ . Show that  $\alpha \in R$ .

- b. If  $R = \mathbb{R}[u, v]/\langle v^2 - u^3 \rangle$  show that  $p(x) = x^2 - u$  has a root over  $K$  but not over  $R$ .
2. Consider a group homomorphism  $\psi : \mathbb{Z}^n \rightarrow \mathbb{Z}^m$ .
  - a. Show that  $\ker(\psi) = 0$  or is isomorphic to  $\mathbb{Z}^k$  where  $n - m \leq k \leq n$ .
  - b. For the specific homomorphism  $\psi : \mathbb{Z}^3 \rightarrow \mathbb{Z}^3$  given by

$$\psi \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 4 & 1 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

show that the cokernel  $\mathbb{Z}^3/\psi(\mathbb{Z}^3)$  is a cyclic group of order 12.

3. Show there exists a nonabelian group  $G$  of order 21. Which Sylow subgroups of  $G$  are normal? How many elements of order three does  $G$  have?
4. Let  $A$  be an  $n \times n$  complex matrix. Show there exist  $n \times n$  matrices  $D$  and  $N$  such that  $A = D + N$  and the following conditions are satisfied:

- $D$  is diagonalizable, i.e., it can be diagonalized after a suitable change of basis;
- $N$  is nilpotent, i.e., some power of  $N$  is zero;
- $D$  and  $N$  commute.

*Hint:* An example of such a decomposition is

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

- Let  $K$  denote the splitting field of the polynomial  $x^5 - 2$  over  $\mathbb{Q}$ .
  - Show that  $x^5 - 1$  splits over  $K$ .
  - Compute the degree  $[K : \mathbb{Q}]$ .
  - Show that  $\text{Gal}(K/\mathbb{Q})$  is not abelian.
- Consider the ideal

$$J = \langle x - t^2, y^2 - t^3 \rangle \subset \mathbb{Q}[x, y, t].$$

Show that the intersection

$$J \cap \mathbb{Q}[x, y]$$

is generated by  $x^3 - y^4$  as an ideal over  $\mathbb{Q}[x, y]$ .