

Mordell-Weil Problem for Cubic Surfaces, Numerical Evidence

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ABSTRACT. Let V be a plane smooth cubic curve over a finitely generated field k . The Mordell-Weil theorem for V states that there is a finite subset $P \subset V(k)$ such that the whole $V(k)$ can be obtained from P by drawing secants and tangents through pairs of previously constructed points and consecutively adding their new intersection points with V . In this paper we present numerical data regarding the analogous statement for cubic surfaces. For the surfaces examined, we also test Manin's conjecture relating the asymptotics of rational points of bounded height on a Fano variety with the rank of the Picard group of the surface.

1. Introduction

Let V be a smooth cubic surface over a field k in \mathbb{P}^3 . If $x, y, z \in V(k)$ are three points (with multiplicities) lying on a line in \mathbb{P}^3 not belonging to V , we write $x = y \circ z$. Thus \circ is a partial and multivalued composition law on $V(k)$. Note that $x \circ x$ is defined as the set of points in the intersection of $V(k)$ with the tangent plane at x . If x does not lie on a line, this is a cubic curve $C(x)$ with double point $x \in V(k)$. This whole set must be considered as the domain of the multivalued expression $x \circ x$, because geometrically all its points can be obtained by drawing tangents with k -rational direction to x . This means that an important source for generating new rational points on the cubic surface will be doubling the points that were already generated. The analogue of the Mordell-Weil theorem for cubic surfaces states that $(V(k), \circ)$ is finitely generated, i.e., there is a finite subset $P \subset V(k)$ such that the whole $V(k)$ can be obtained from P by drawing secants and tangent planes through pairs of (not necessarily distinct) previously constructed points, and consecutively adding their new intersection points with V . By drawing secants we can add only one rational point to P , while tangent sections give us an infinite number of points that can be generated, by the note above. For a more thorough discussion of various versions of finite generation cf. [KM01]. Note that, by Theorem 11.7 of [Man86], finite generation of $(V(k), \circ)$ implies that the universal quasi-group of $(V(k), \circ)$, as defined in [Man86], chapter II, is finite and has $2^n 3^m$ elements for some $n, m \in \mathbb{Z}_{\geq 0}$.

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In the following, we present the procedure we used to test whether $(V(\mathbb{Q}), \circ)$ is finitely generated, and the results we obtained for thirteen diagonal cubic surfaces, six of them having the rank of their Picard group equal to 1, and seven of them mentioned in [PT01], illustrating the cases of surfaces with ranks 2 and 3 of the Picard group. We also bring numerical evidence supporting Manin's conjecture for the asymptotics of rational points of bounded height on a Fano variety. Note that John Slater and Sir Peter Swinnerton-Dyer have proved in [SP98] a one-sided estimate for the conjecture in the case when V contains two rational skew lines. All the computations were done using the Magma computer algebra system (cf. [BCP97].)

2. Description of the procedure

Let $ax^3 + by^3 + cz^3 + du^3 = 0$, where a, b, c, d are nonzero integers, be a diagonal cubic surface. Using a program due to Dan Bernstein (see [Ber01]), we find all rational points on this surface up to height $H = 10^5$ or $H = 1.5 \cdot 10^5$, where the height of a rational point $P = (x : y : z : u)$, with $x, y, z, u \in \mathbb{Z}$ and $\gcd(x, y, z, u) = 1$ is defined as

$$h_{\max}(P) := \max\{|x|, |y|, |z|, |u|\}.$$

We consider also another height function $h_{\text{sum}} : V(\mathbb{Q}) \rightarrow \mathbb{R}_+$ defined by

$$h_{\text{sum}}(P) := |x| + |y| + |z| + |u|.$$

Note that a rational point P can be uniquely written in the above form up to a sign change of the coordinates. So, if we assume, in addition, that the first nonzero coordinate of P is positive, then there is a unique such 'canonical' form corresponding to each point P . We order the rational points by increasing h_{sum} . If there are two or more points having the same height h_{sum} , then we order them lexicographically according to their coordinates in the canonical form. This defines a total order on the set of rational points. We will write $P < Q$ if P precedes Q in the sorted list, and use the number of a point in this list as its name. We will also refer to this number as the *index* of a rational point.

We will use the h_{\max} height function only to study the asymptotics of the number of rational points on a cubic surface, while for the ordering of the points and in the implementation of the main function we will use h_{sum} .

For testing whether a given set of rational points is generating, we use the procedure *Test Generating Set (TGS)*, which is described below.

The procedure implements essentially a descent method. Given an index bound n and a set of points *GeneratedSet* that is presumably generating, we perform the following iterative process. In one iteration of loop, we consider all points in the range $\{1, \dots, n\}$ that are not in *GeneratedSet* and test whether they can be decomposed as $x \circ y$, with $x, y \in \text{GeneratedSet}$. Every point that can be decomposed in such a way is added to the *GeneratedSet* and at the end of the loop, the procedure is reiterated. As now *GeneratedSet* is bigger, there may be additional points in the range $\{1, \dots, n\}$ that can be generated because we can choose the points x, y for a possible decomposition from a bigger set. The procedure is repeated until *GeneratedSet* stabilizes, i.e., until some iteration of the loop does not add any new points to the *GeneratedSet*.

In order to avoid repeating some operations of composing points, we use the additional variables *OldGeneratedSet*, *JustAdded* and *Decomp. OldGeneratedSet*

stores the value of *GeneratedSet* at the beginning of the iteration of the loop. At the end of the preceding loop, a number of points will have been added to *GeneratedSet*. These points are stored in the set variable *JustAdded*. During an iteration of the loop, we store in *Decomp* decompositions of the type $i = j \circ k$, with $i, j, k \leq n$, $i, k \notin \text{GeneratedSet}$ and $j \in \text{GeneratedSet}$. These are the only decompositions that we could further use. Indeed, if, at some point, k was added to *GeneratedSet*, then by searching in *Decomp*, we would find the decomposition $j \circ k$ of i and we would add i to *GeneratedSet* without performing any composition of points (which requires multiplications, so is computationally expensive) because we know, by the way we constructed *Decomp*, that $j \in \text{GeneratedSet}$ already.

Receiving as input the parameters *GeneratedSet* (a set of points in $V(\mathbb{Q})$ that is assumed to be generating), and n (the index bound for the points used in the decompositions), the *TGS* procedure does the following:

- (1) Set $\text{Decomp} = \emptyset$, $\text{OldGeneratedSet} = \emptyset$.
- (2) Set $\text{JustAdded} = \text{GeneratedSet} \setminus \text{OldGeneratedSet}$,
 $\text{OldGeneratedSet} = \text{GeneratedSet}$.
- (3) If $\text{JustAdded} = \emptyset$, return *GeneratedSet*.
- (4) For every point $i \in \{1, 2, \dots, n\} \setminus \text{GeneratedSet}$ do:
 search in *Decomp* for decompositions of i as $x \circ y$ with $y \in \text{JustAdded}$
 if such a decomposition exists, add i to *GeneratedSet*
 else for every point j in *JustAdded* do:
 $k = i \circ j$
 if $k \in \text{JustAdded}$
 add i to *GeneratedSet*
 break
 else if $k \leq n$ add the decomposition $(j \circ k)$ of i to *Decomp*
 end for
 end for.
- (5) Go to step 2.

Let us explain in more detail the way the algorithm works. Suppose that an iteration of the outer loop has just finished, and we are in step 2. We set $\text{JustAdded} = \text{GeneratedSet} \setminus \text{OldGeneratedSet}$ and test whether this is the empty set. If this is so, then during the last iteration we could not generate any new points, so the maximum set of points that can be generated is the current *GeneratedSet*. If *JustAdded* is not empty, then during the last iteration we found a number of new points that could be generated and added them to *GeneratedSet* (these are the elements of *JustAdded*), so there is hope of generating other points. We consider a point $i \notin \text{GeneratedSet}$. Since we have already tested during the previous iteration whether we could decompose i as $x \circ y$, with $x, y \in \text{OldGeneratedSet}$, all we have to check now is whether we can write $i = x \circ y$ for $x \in \text{JustAdded}$ and either $y \in \text{OldGeneratedSet}$ or $y \in \text{JustAdded}$. At the previous iterations of the loop all compositions of i with points in *OldGeneratedSet* that could further be used (i.e., compositions whose result is not bigger than n) were stored in *Decomp*, so we can check for the first possibility by searching in the vector *Decomp*. Since by construction we only store in *Decomp* decompositions of the type $x \circ y$, with $x \in \text{GeneratedSet}$, all we have to check in the beginning of step 4 is whether $y \in \text{JustAdded}$ —we are sure that $x \in \text{GeneratedSet}$. In order to check for the second possibility, we have to compose i with every point $j \in \text{JustAdded}$. If the result k of

the composition is in *JustAdded*, then we can write x as a composition of two points in *JustAdded*, so we add i to *GeneratedSet*. If the result $k \notin \text{JustAdded}$, but could be further used (i.e., $k \leq n$), then we store the corresponding decomposition $j \circ k$ of i in *Decomp*. The ‘out of bounds’ compositions, i.e., such that $i \circ j > n$, are implicitly remembered in the process (in the sense that they are done only once.)

Using the vector *Decomp* of course implies a trade off between space and speed, but we considered the latter to be more important. Even with *Decomp*, the computations for *TGS* for bounds n in the range of 10^5 last for several days and sometimes even weeks on an Intel Pentium IV processor with 2.26 GHz.

Before we proceed with the presentation of the results, let us provide an estimate of the height of the composition of two rational points. Here by h we mean either h_{\max} or h_{sum} since the estimation of the asymptotics does not depend on the choice of the height function.

LEMMA 2.1. *Let $V : ax^3 + by^3 + cz^3 + du^3 = 0$ be a diagonal cubic surface, where a, b, c, d are nonzero integers, and let $K := \max\{|a|, |b|, |c|, |d|\}$. If A_1 and A_2 are two distinct points in $V(\mathbb{Q})$ that do not lie on a line in V , then*

$$h(A_1 \circ A_2) = O(K \cdot h(A_1)^2 \cdot h(A_2)^2).$$

Proof: Let $A_1 = (x_1 : y_1 : z_1 : u_1)$, $A_2 = (x_2 : y_2 : z_2 : u_2)$ be in canonical form. Then one can check that

$$A_1 \circ A_2 = (\alpha x_1 - \beta x_2 : \alpha y_1 - \beta y_2 : \alpha z_1 - \beta z_2 : \alpha u_1 - \beta u_2),$$

where

$$\begin{aligned} \alpha &= ax_1x_2^2 + by_1y_2^2 + cz_1z_2^2 + du_1u_2^2 \in \mathbb{Z}, \\ \beta &= ax_1^2x_2 + by_1^2y_2 + cz_1^2z_2 + du_1^2u_2 \in \mathbb{Z}. \end{aligned}$$

Since the above coordinates of $A_1 \circ A_2$ are integers, the conclusion follows. This upper bound cannot be improved because, in most cases, the formula given represents $A_1 \circ A_2$ in canonical form (up to a sign change of the coordinates).

Concerning the doubling of points, if $A \in V(Q)$ is a rational point not lying on a line in V , then there is no upper bound for the height of the points in $A \circ A$ (since there are infinitely many such points). On the other hand, there can be many points of small height in $A \circ A$, especially if A has small height.

3. Results

Listed below are the thirteen diagonal cubic surfaces that were tested for finite generation, ordered according to the ranks of their Picard groups:

Rank 1 Picard group:

- (1) $x^3 + 2y^3 + 3z^3 + 4u^3 = 0$.
- (2) $x^3 + 2y^3 + 3z^3 + 5u^3 = 0$.
- (3) $17x^3 + 18y^3 + 19z^3 + 20u^3 = 0$.
- (4) $4x^3 + 5y^3 + 6z^3 + 7u^3 = 0$.
- (5) $9x^3 + 10y^3 + 11z^3 + 12u^3 = 0$.
- (6) $x^3 + 5y^3 + 6z^3 + 10u^3 = 0$.

Rank 2 Picard group:

$$(7) \quad x^3 + y^3 + 2z^3 + 4u^3 = 0.$$

$$(8) \quad x^3 + y^3 + 5z^3 + 25u^3 = 0.$$

$$(9) \quad x^3 + y^3 + 3z^3 + 9u^3 = 0.$$

Rank 3 Picard group:

$$(10) \quad x^3 + y^3 + 2z^3 + 2u^3 = 0.$$

$$(11) \quad x^3 + y^3 + 5z^3 + 5u^3 = 0.$$

$$(12) \quad x^3 + y^3 + 7z^3 + 7u^3 = 0.$$

$$(13) \quad 2x^3 + 2y^3 + 3z^3 + 3u^3 = 0.$$

The first six cubic surfaces illustrate the case of Picard group of rank 1. The third surface was considered as an example of a diagonal cubic surface with bigger coefficients. The lack of success in finding a generating set for this surface (as opposed to all the other surfaces examined by that point) motivated the study of the surfaces 4–5, which have coefficients of intermediate value between the coefficients of the first, successful surface, and the third, problematic one. Surface 6 is aimed to illustrate the case of surfaces with ‘random’ coefficients. The remaining seven surfaces were taken from [PT01] as examples of cubic surfaces with the ranks of the Picard group 2 and 3.

In order to find a suitable generating set G to begin with, we tested several small sets for finite generation up to a small index n ($n = 100$, or $n = 1000$). We observed that, if the set G generates more than 80% – 90% of the first n points for a small n , then this is a good indicator that the set G will generate roughly the same percentage of all points up to a much bigger index bound N (which we took to be either $5 \cdot 10^4$ or 10^5). We chose the initial small sets to be the set of points of indices $\{1, 2, 3, 4\}$. If this did not yield a large enough percentage of points generated, we would enlarge the initial set to $G = \{1, 2, 3, 4, 5\}$, and continue this way. Generally, we were ‘lucky’, in the sense that a few tries would provide us with a good generating set G (a set G that generates most of the first n points.) Then we would eliminate from G the ‘superfluous’ points, i.e., the points that could be obtained by composing other points in G . This is the reason for which, for example, the first surface has $G = \{3\}$ instead of $G = \{1, 2, 3, 4\}$: the points of indices 1, 2 and 4 lie in the tangent plane at the point of index 3.

At first, the only exception was the surface 3, which represents, at least computationally, a problem. Having added the surfaces 4–5, we noticed that it is hard to find a generating set using this naive method for these surfaces as well.

The generating sets we found are given in Table 1; these are listed both as sets of indices and as sets of rational points. Here, and in all subsequent tables, the label ‘ S ’ stands for ‘surface’.

Before we go on and list the results we obtained using the *TestGeneratingSet* procedure, let us provide an indication of the asymptotics of the number of points on each cubic surface up to some height H . Note that, as we used Dan Bernstein’s program to find rational points on the diagonal cubic surfaces, here ‘height’ refers to h_{\max} . The asymptotics of the number of points seems to be related to the percentage of points that can be generated up to some height. For the last seven surfaces, we did not take into consideration the points on the trivial rational lines, i.e., points of the type $(x : -x : y : -y)$, except for the point $(1 : -1 : 0 : 0)$ on the surfaces 7–9 and the points $(1 : -1 : 0 : 0)$, $(0 : 0 : 1 : -1)$, $(1 : -1 : 1 : -1)$ and $(1 : -1 : -1 : 1)$ on the surfaces 10–13, which we need for finite generation.

TABLE 1. Generating sets

S	G as set of indices	G as set of points
1	{3}	{(1 : -1 : -1 : 1)}
2	{1, 2, 4}	{(0 : 1 : 1 : -1), (1 : 1 : -1 : 0), (2 : -2 : 1 : 1)}
6	{2}	{(1 : -1 : -1 : 1)}
7	{3}	{(1 : -1 : -1 : 1)}
8	{1, 2}	{(1 : -1 : 0 : 0), (1 : 4 : -2 : -1)}
9	{1, 2, 4}	{(1 : -1 : 0 : 0), (1 : 2 : 0 : -1), (1 : 2 : -3 : 2)}
10	{5, 6}	{(1 : -1 : -1 : 1), (1 : -1 : 1 : -1)}
11	{3, 4}	{(1 : -1 : -1 : 1), (1 : -1 : 1 : -1)}
12	{1, 2, 5, 6}	{(0 : 0 : 1 : -1), (1 : -1 : 0 : 0), (1 : -1 : -1 : 1), (1 : -1 : 1 : -1)}
13	{1, 2, 3, 4, 5}	{(0 : 0 : 1 : -1), (1 : -1 : 0 : 0), (1 : -1 : -1 : 1), (1 : -1 : 1 : -1), (3 : -6 : 1 : 5)}

TABLE 2. Data on points of bounded height

S	Number of points up to height									
	100	200	500	1000	2000	5000	10000	20000	50000	100000
1	77	163	436	906	1827	4408	8754	17332	43280	86329
2	180	358	855	1683	3244	8097	16436	32704	82581	166825
3	16	25	62	117	204	502	1055	2084	5479	10840
4	37	78	206	414	778	1937	3877	7756	19701	39433
5	37	67	165	310	595	1580	3148	6257	15499	31134
6	55	120	316	646	1285	3131	6397	12753	32072	64102
7	196	458	1308	2746	6004	16758	35958	75984	205284	433526
8	142	292	766	1734	3872	10892	23338	49608	135128	286040
9	200	438	1270	2768	6200	17434	37018	78980	215626	455164
10	666	1630	5410	12870	29926	89218	205198	465226	1364810	3051198
11	412	1012	3328	7964	18676	56412	131512	299776	881774	1976482
12	702	1870	6010	14130	33156	100580	228696	520700	1526532	3420784
13	384	1052	3196	7752	18400	56348	130476	298860	876776	1966160

Table 2 includes intermediate results of the number of points up to different height limits. These results seem to confirm Manin’s conjecture relating the asymptotics of rational points of bounded height on a Fano variety with the rank of the Picard group of the surface (see [FMT89]):

$$\#\{P \in V(\mathbb{Q}) : h(P) < H\} \sim CH \log^{\text{rkPic}(V)-1} H$$

for $H \rightarrow \infty$, where h is an anticanonical height on V .

For the surfaces with rank of the Picard group equal to 1 we computed, additionally, the number of rational points up to slightly greater height limits, as summarized in Table 3 (‘-’ means ‘not computed’).

Relevant to our claim that these results seem to confirm Manin’s conjecture are the graphs (Figures 1, 2, and 3) based on the tables above. In all graphs, we plotted the number of points up to height H divided by $H \log^{\text{rkPic}(V)-1} H$ for various values of H . The conjecture would be verified if the plotted points would become arbitrarily close, in the limit, to a line parallel to the Ox axis, of equation $y = C$, where C is the constant predicted by Manin’s conjecture. For a conjecture about the value of this constant, see [PT01].

TABLE 3. Additional data for surfaces with Picard group of rank one

S	Number of points up to height			
	150000	200000	250000	300000
1	129473	—	—	—
2	250286	—	—	—
3	16123	21627	27026	32507
4	59100	78498	—	—
5	46436	61958	77518	93079
6	96065	—	—	—

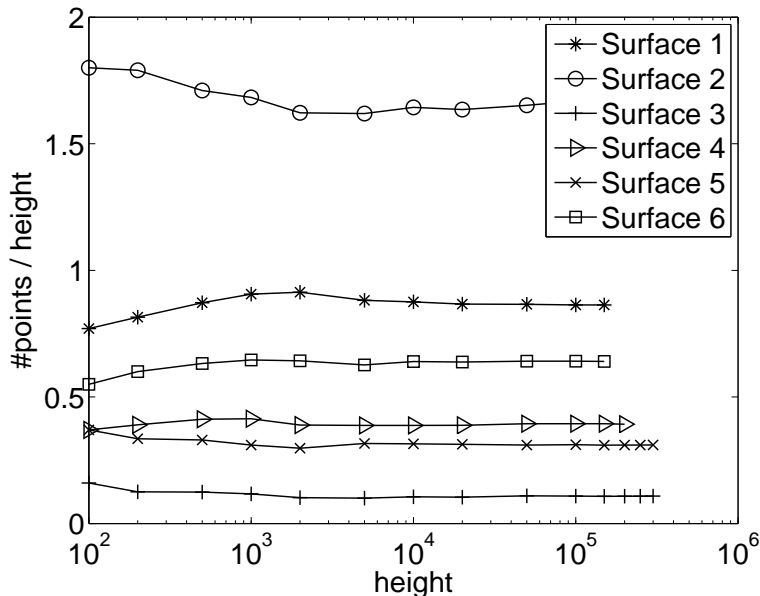


FIGURE 1. Surfaces with Picard group of rank 1

In the remaining, by ‘height’ we mean h_{sum} .

Note that for the surfaces with rank of the Picard group equal to two, most of the points are ‘doubled’, i.e., if $(x : y : z : u)$ is a point on the cubic surface, then so is $(y : x : z : u)$, while for the surfaces with rank of the Picard group equal to three, most of the points are ‘quadrupled’, i.e., if $(x : y : z : u)$ is a point on the cubic surface, then so are $(y : x : z : u)$, $(x : y : u : z)$ and $(y : x : u : z)$. In the following we list the results which were obtained using the *TestGeneratingSet* procedure. The generating sets used are the ones enumerated above, while the index bounds and the corresponding height bounds are given in the third and second columns of Table 4. ‘# iter’ is the number of iterations of the outer loop of the procedure, and the ‘first bad point’ refers to the point of smallest index that could not be generated by the procedure. For example, the first line in the table reads “The procedure *TestGeneratingSet* called for surface 1, with index bound 100 corresponding to the

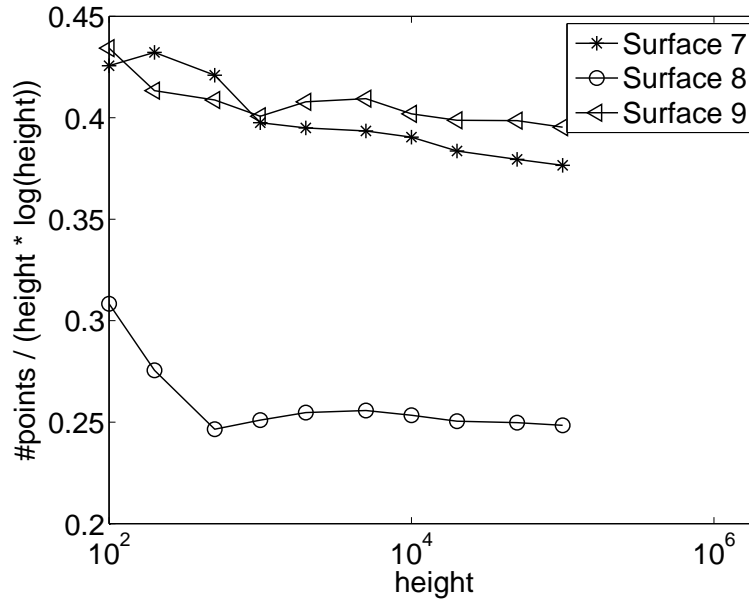


FIGURE 2. Surfaces with Picard group of rank 2

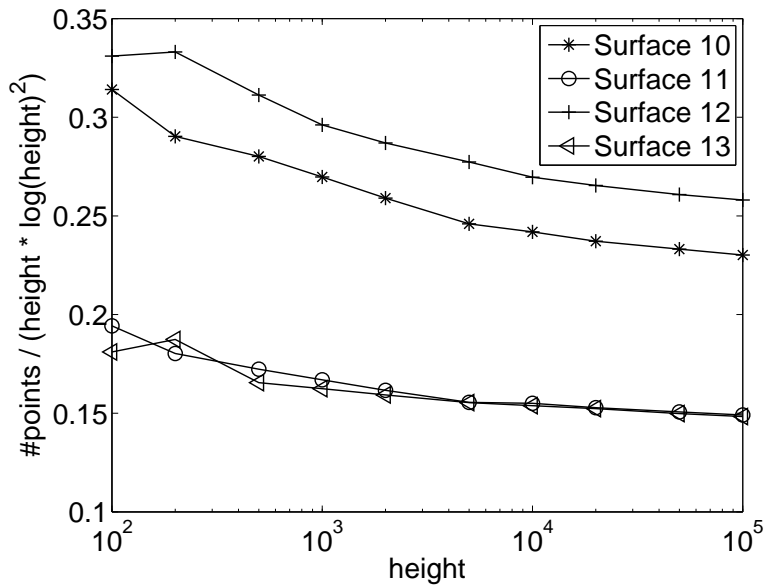


FIGURE 3. Surfaces with Picard group of rank 3

height bound 317, and initial generating set $G = \{3\}$ (or $G = \{(1 : -1 : -1 : 1)\}$), generates 74 rational points, which represents 74.0% of the first 100 points, in 4 iterations of the outer loop. The point of smallest height which could not be generated has index 30 and height 86.”

TABLE 4. Statistics on the points generated by our potential generating set

Surface	Height bound	Index bound	# points generated	% points generated	# iter	First bad point	
						Index	Height
1	317	100	74	74.0	4	30	86
1	617	200	160	80.0	9	30	86
1	1,443	500	463	92.6	16	42	130
1	2,788	1,000	923	92.3	15	255	788
1	5,574	2,000	1,859	93.0	14	543	1,541
1	14,456	5,000	4,747	94.9	15	1,145	3,192
1	29,074	10,000	9,462	94.6	14	1,593	4,423
1	58,775	20,000	18,957	94.8	14	3,633	10,322
1	147,343	50,000	47,418	94.8	13	8,522	24,677
1	296,822	100,000	94,910	94.9	13	8,522	24,677
2	150	100	97	97.0	7	85	124
2	282	200	196	98.0	9	90	134
2	703	500	483	96.6	8	258	364
2	1,477	1,000	973	97.3	9	358	511
2	3,020	2,000	1931	96.6	9	625	943
2	7,663	5,000	4,813	96.3	10	1,040	1,542
2	15,405	10,000	9,659	96.6	11	1,775	2,656
2	30,651	20,000	19,259	96.3	11	4,262	6,539
2	75,845	50,000	48,181	96.3	11	10,073	15,539
2	151,171	100,000	96,477	96.5	12	15,223	23,243
6	388	100	86	86.0	5	49	209
6	762	200	176	88.0	5	49	209
6	1,864	500	468	93.6	10	169	641
6	3,687	1,000	937	93.7	11	181	688
6	7,557	2,000	1,867	93.3	11	513	1,926
6	18,976	5,000	4,677	93.6	11	1,078	3,984
6	37,612	10,000	9,410	94.1	11	2,271	8,661
6	74,617	20,000	18,963	94.8	11	2,662	10,125
6	186,532	50,000	47,436	94.9	12	6,373	24,068
7	129	100	100	100.0	6	—	—
7	245	200	194	97.0	6	127	167
7	538	500	490	98.0	8	304	376
7	980	1,000	990	99.0	7	550	612
7	1,889	2,000	1,984	99.2	7	1,022	992
7	4,230	5,000	4,974	99.5	7	2,620	2,401
7	7,974	10,000	9,934	99.3	8	5,610	4,707
7	14,775	20,000	19,934	99.7	8	7,512	6,222
7	34,339	50,000	49,880	99.8	7	19,666	14,554
7	64,682	100,000	99,812	99.8	8	38,212	26,672
7	94,215	150,000	149,744	99.8	9	38,212	26,672

TABLE 5. Statistics on the points generated by our potential generating set, contd.

Surface	Height bound	Index bound	# points generated	% points generated	# iter	First bad point	
						Index	Height
8	172	100	81	81.0	6	42	78
8	316	200	170	85.0	8	56	104
8	750	500	488	97.6	9	152	234
8	1,412	1,000	988	98.8	8	516	774
8	2,484	2,000	1,960	98.0	8	516	774
8	5,632	5,000	4,922	98.4	9	1,855	2,322
8	10,354	10,000	9,874	98.7	8	3,708	4,296
8	19,444	20,000	19,836	99.2	8	6,852	7,538
8	44,750	50,000	49,720	99.4	8	16,058	15,812
8	84,436	100,000	99,626	99.6	9	32,420	30,072
9	114	100	48	48.0	4	8	24
9	242	200	198	99.0	9	126	146
9	522	500	484	96.8	9	318	346
9	978	1,000	956	95.6	11	379	414
9	1,822	2,000	1,968	98.4	9	781	770
9	3,878	5,000	4,954	99.1	9	1,602	1,472
9	7,254	10,000	9,936	99.4	9	3,728	3,046
9	13,610	20,000	19,908	99.5	9	10,420	7,522
9	31,320	50,000	49,806	99.6	8	21,142	14,342
9	58,852	100,000	99,778	99.8	9	32,036	20,884
10	61	100	92	92.0	3	79	51
10	91	200	200	100.0	3	—	—
10	214	500	496	99.2	5	419	184
10	358	1,000	980	98.0	6	651	255
10	612	2,000	1,996	99.8	5	1,791	554
10	1,225	5,000	4,940	98.8	5	2,259	674
10	2,143	10,000	9,916	99.2	6	3,675	976
10	3,806	20,000	19,852	99.3	6	5,779	1,396
10	8,020	50,000	49,732	99.5	7	20,870	3,949
11	94	100	89	89.0	5	61	56
11	144	200	184	92.0	5	61	56
11	274	500	492	98.4	6	257	174
11	474	1,000	988	98.8	6	757	382
11	802	2,000	1,960	98.0	6	1,177	528
11	1,688	5,000	4,924	98.5	7	1,495	642
11	2,882	10,000	9,888	98.9	8	3,873	1,386
11	5,100	20,000	19,732	98.7	9	6,207	2,004
11	10,880	50,000	49,544	99.1	9	11,737	3,308

TABLE 6. Statistics on the points generated by our potential generating set, contd.

Surface	Height bound	Index bound	# points generated	% points generated	# iter	First bad point	
						Index	Height
12	48	100	96	96.0	4	95	46
12	92	200	192	96.0	4	95	46
12	186	500	476	95.2	5	223	106
12	286	1,000	956	95.6	6	223	106
12	484	2,000	1,969	98.5	5	964	284
12	1,014	5,000	4,911	98.2	6	2,315	548
12	1,740	10,000	9,880	98.8	6	3,486	764
12	3,066	20,000	19,832	99.2	7	4,030	856
12	6,514	50,000	49,532	99.1	7	16,064	2,578
13	106	100	96	96.0	4	41	75
13	167	200	196	98.0	5	169	153
13	316	500	484	96.8	6	169	153
13	515	1,000	980	98.0	6	572	360
13	910	2,000	1,944	97.2	6	860	465
13	1,885	5,000	4,896	97.9	7	1,937	897
13	3,310	10,000	9,780	97.8	7	3,102	1,323
13	5,727	20,000	19,672	98.4	7	4,785	1,816
13	12,139	50,000	48,256	96.5	8	8,202	2,805

Note that, in general, when using a greater index bound we found that the ‘first bad point’ changed (i.e., another point of greater height and index became the ‘first bad point’), meaning that using stepping stones of bigger height typically fills up the gaps obtained when using a lower index bound. This is a good indicator that if we continue increasing the index (and thus the height) bounds, we will gradually generate *all* the points up to bigger and bigger heights.

Let us see now what happens with the ‘problematic surfaces’ 3–5. The data is displayed in Table 7. Unfortunately, any try of finding a generating set to begin with, that finds ‘first bad points’ of increasing height, and that generates a percentage of points similar to the ones obtained for the ‘good’ surfaces was not successful. Not even a ‘brute force’ approach like considering the initial *GeneratedSet* to be, say, the first 100 or 1000 points does not yield satisfactory results. The results are better for the surfaces 4–5 than for the surface 3, with the biggest coefficients, but still very ‘bad’. In Table 7 we provide an illustration of the behavior of these surfaces when starting with the $GeneratedSet = \{1, 2, \dots, 10\}$.

These results seem to support either that $\{1, 2, \dots, 10\}$ is not a generating set for any of the three surfaces, or that the stepping stones needed to fill up the gaps (i.e., the rational points needed to decompose the ‘first bad points’) have very big heights. Although the percentages of generated points obtained for the surfaces 4–5 are slightly better than the percentages for the surface 3, they still become smaller and smaller as the index bound limit (and so also the height) grow. But the most important negative indicator is that ‘the first bad point’ never changes.

TABLE 7. Statistics for the ‘problematic surfaces’

Surface	Height bound	Index bound	# points generated	% points generated	# iter	First bad point	
						Index	Height
3	2,161	100	17	17.0	2	13	203
3	5,495	200	24	12.0	2	13	203
3	13,429	500	35	7.0	2	13	203
3	25,874	1,000	49	4.9	2	13	203
3	51,663	2,000	81	4.1	2	13	203
3	124,062	5,000	154	3.1	2	13	203
3	251,103	10,000	274	2.7	2	13	203
3	505,619	20,000	429	2.1	2	13	203
4	658	100	26	26.0	1	12	50
4	1,345	200	50	25.0	2	12	50
4	3,307	500	102	20.4	2	12	50
4	6,774	1,000	172	17.2	3	12	50
4	13,772	2,000	284	14.2	3	12	50
4	34,552	5,000	487	9.7	3	12	50
4	68,425	10,000	781	7.8	3	12	50
4	135,691	20,000	1,222	6.1	4	12	50
5	844	100	19	19.0	1	13	103
5	1,691	200	26	13.0	2	13	103
5	4,394	500	51	10.2	2	13	103
5	8,780	1,000	80	8.0	2	13	103
5	16,962	2,000	119	6.0	2	13	103
5	43,224	5,000	216	4.3	2	13	103
5	87,176	10,000	338	3.4	3	13	103
5	174,128	20,000	538	2.7	3	13	103

In order to make progress, we introduced another approach to finding a generating set for the surfaces 3–5, based on the idea of ‘throwing in’ (adding to the *Generated Set*) the first bad points if they cannot be generated by decomposition. Our aim is to obtain, after adding sufficiently many ‘first bad points’, a set of points that generates a stable (or even better, increasing) percentage of points for increasing index bounds, and a ‘changing first bad point’ behavior, i.e., applying the *TGS* procedure to increasing index bounds would result in finding ‘first bad points’ of increasing heights.

We implement this new approach in the following way. We apply the *TGS* procedure to a (small) generating set and an index bound of 1000. We obtain a ‘first bad point’ that unfortunately stays the same when increasing the index bound (as observed when using our first approach). We apply again the *TGS* procedure to the initial generating set *and* this first bad point, with an index bound of 1000. We obtain another ‘first bad point’, of bigger index and height than the initial one. We add this point to our generating set (which now contains also the initial ‘first bad point’) and continue this way. We stop when we have added sufficiently many ‘first bad points’ to our initial set so that this new, bigger generating set fulfills the

two objectives mentioned above. Once we have obtained such a set, we stop adding points to our generating set and just increase the index bounds to make sure the percentage of generated points is indeed stable or increasing, and that the height of the ‘first bad point’ grows as the index bound is increased.

For example, for surface 4, we start with *Generated Set* = $\{1, 2, \dots, 10\}$. We obtain the first bad point 12, which is stable—stays the same even if we increase the index bound. We add it to the *Generated Set* and call again the *TGS* procedure. We obtain more points, and another first bad point. We add this new bad point to the *Generated Set* and continue this way, gradually filling the holes. At first we kept the index bound constant, until we obtained a reasonable percentage of generated points. Then we tested whether the ‘first bad point’ changes when increasing the index bound and keeping the initial *Generated Set* constant (i.e., we stopped filling the holes, and just increased the index bound.) For surfaces 4 and 5 this approach seems reasonably successful, as reflected in Tables 8 and 9.

Unfortunately, for surface 3 this approach does not seem to work. After adding many more ‘first bad points’ to the initial generating set than for the surfaces 4–5, we still did not obtain a ‘good’ generating set, as illustrated in Table 10.

Since this process was too slow, we tried ‘throwing’ in our *Generated Set* not only the first bad point, but the first 10 bad points at every step (see Table 11).

This was again too slow, so we started inserting the first 20 bad points to our *Generated Set* (see Table 12).

Next we present other statistical data.

It seems that the percentage of points on a surface that can be strongly decomposed (a point x is *strongly decomposable* if it has a decomposition $x = y \circ z$ with $y, z < x$) up to some index N is approximately constant for various values of N . This suggests that this percentage may be an invariant for the surface.

It seems likely that if this percentage is bigger then *TestGeneratingSet* will generate more points (up to some index), using a suitable *GeneratedSet*. This is confirmed if we study the first two surfaces. Surface 1 has roughly $\frac{N}{8}$ points that are not strongly decomposable up to the index N (for $N \geq 1000$), while the surface 2 has only $\sim \frac{N}{11}$ such points; and indeed, if we compare the results of *TGS* for the two surfaces, we notice that *TGS* for the surface 2 generates more points (up to the same index) than *TGS* for the surface 1. Also, note that the percentage of points that are strongly decomposable for the surface 3 is very small (approximately 10%.) This may be one of the explanations for our lack of success with this surface.

4. Conclusion

The theory surrounding the Mordell-Weil problem for cubic surfaces seems not very well developed, mainly because of the difficulties caused by the lack of a group structure on the operation of composing points. In this paper we presented numerical data for thirteen diagonal cubic surfaces, in the hope of developing some intuition on a possible finiteness conjecture (first mentioned by Manin, cf. [Man86] and [Man97]). For each of the surfaces, we tried to find a generating set. A naive method gave positive results for ten of the surfaces, while a more rigorous method was needed to obtain similar (but not as positive) results for two of the other surfaces. For these surfaces, the numerical data suggest that they might be indeed finitely generated. The remaining surface resisted both methods. We cannot say,

TABLE 8. Analysis of surface 4 using ‘TGS’ procedure

Surface	Height bound	Index bound	# points generated	% points generated	# iter	First bad point	
						Index	Height
4	6,774	1,000	172	17.2	3	12	50
4	6,774	1,000	177	17.7	3	13	55
4	6,774	1,000	194	19.4	4	14	63
4	6,774	1,000	210	21.0	4	15	73
4	6,774	1,000	218	21.8	4	20	107
4	6,774	1,000	230	23.0	4	21	108
4	6,774	1,000	237	23.7	4	22	110
4	6,774	1,000	249	24.9	5	23	125
4	6,774	1,000	268	26.8	6	25	179
4	6,774	1,000	282	28.2	6	27	193
4	6,774	1,000	296	29.6	6	28	199
4	6,774	1,000	325	32.5	13	32	215
4	6,774	1,000	328	32.8	13	35	249
4	6,774	1,000	335	33.5	13	37	262
4	6,774	1,000	338	33.8	13	43	297
4	6,774	1,000	342	34.2	13	49	317
4	6,774	1,000	349	34.9	13	52	329
4	6,774	1,000	351	35.1	13	58	370
4	6,774	1,000	353	35.3	13	62	396
4	6,774	1,000	360	36.0	13	66	413
4	6,774	1,000	372	37.2	13	69	438
4	6,774	1,000	394	39.4	18	73	467
4	6,774	1,000	400	40.0	18	76	487
4	34,552	5,000	1,331	26.6	38	89	570
4	68,425	10,000	2,769	27.7	50	92	611
4	135,691	20,000	6,365	31.8	53	189	1,230
4	204,042	30,000	10,142	33.8	50	233	1,605
4	271,092	40,000	14,403	36.0	45	324	2,115
4	339,994	50,000	18,409	36.8	51	352	2,387

however, whether this means that the surface is not finitely generated, or that this is just a sign of the limits of the methods used.

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TABLE 9. Analysis of surface 5 using 'TGS' procedure

Surface	Height bound	Index bound	# points generated	% points generated	# iter	First bad point	
						Index	Height
5	8,780	1,000	80	8.0	2	13	103
5	8,780	1,000	87	8.7	3	14	111
5	8,780	1,000	100	10.0	3	15	112
5	8,780	1,000	114	11.4	6	16	122
5	8,780	1,000	142	14.2	8	17	125
5	8,780	1,000	149	14.9	8	18	126
5	8,780	1,000	157	15.7	8	19	127
5	8,780	1,000	170	17.0	8	21	150
5	8,780	1,000	175	17.5	8	23	168
5	8,780	1,000	177	17.7	8	25	177
5	8,780	1,000	207	20.7	16	27	188
5	8,780	1,000	211	21.1	16	28	190
5	8,780	1,000	219	21.9	16	32	211
5	8,780	1,000	223	22.3	16	37	276
5	8,780	1,000	230	23.0	16	39	298
5	8,780	1,000	232	23.2	16	44	350
5	8,780	1,000	236	23.6	16	45	363
5	8,780	1,000	237	23.7	16	46	367
5	8,780	1,000	242	24.2	16	47	369
5	8,780	1,000	268	26.8	16	56	427
5	8,780	1,000	276	27.6	16	57	431
5	8,780	1,000	282	28.2	16	59	445
5	8,780	1,000	311	31.1	16	60	464
5	8,780	1,000	313	31.3	16	62	487
5	8,780	1,000	319	31.9	16	66	581
5	8,780	1,000	337	33.7	16	68	595
5	8,780	1,000	339	33.9	16	69	602
5	8,780	1,000	347	34.7	16	75	631
5	8,780	1,000	356	35.6	16	76	637
5	8,780	1,000	365	36.5	16	84	695
5	8,780	1,000	369	36.9	16	87	719
5	8,780	1,000	380	38.0	16	88	733
5	8,780	1,000	385	38.5	16	91	745
5	8,780	1,000	390	39.0	16	93	771
5	8,780	1,000	409	40.9	16	96	801
5	8,780	1,000	413	41.3	16	103	862
5	43,224	5,000	1,881	37.6	29	118	1,015
5	87,176	10,000	3,650	36.5	32	145	1,197
5	174,128	20,000	7,236	36.2	37	295	2,554
5	262,052	30,000	11,367	37.9	44	325	2,774
5	349,121	40,000	15,842	39.6	44	461	3,988
5	437,046	50,000	20,103	40.2	35	461	3,988

TABLE 10. Surface 3 remains problematic after adding the ‘first bad point’ at every step

Surface	Height bound	Index bound	# points generated	% points generated	# iter	First bad point	
						Index	Height
3	25,874	1,000	49	4.9	2	13	203
3	25,874	1,000	52	5.2	2	14	248
3	25,874	1,000	54	5.4	2	15	260
3	25,874	1,000	57	5.7	2	16	264
3	25,874	1,000	60	6.0	2	18	335
3	25,874	1,000	62	6.2	2	19	337
3	25,874	1,000	64	6.4	2	20	383
3	25,874	1,000	66	6.6	2	21	413
3	25,874	1,000	67	6.7	2	22	433
3	25,874	1,000	69	6.9	2	23	434
3	25,874	1,000	71	7.1	2	26	526
3	25,874	1,000	73	7.3	2	27	573
3	25,874	1,000	76	7.6	2	28	605
3	25,874	1,000	77	7.7	2	29	630
3	25,874	1,000	78	7.8	2	31	699
3	25,874	1,000	80	8.0	2	32	711
3	25,874	1,000	82	8.2	2	35	754
3	25,874	1,000	85	8.5	2	36	772
3	25,874	1,000	86	8.6	2	37	775
3	25,874	1,000	88	8.8	2	39	808
3	25,874	1,000	90	9.0	2	40	819
3	25,874	1,000	93	9.3	2	41	853
3	25,874	1,000	95	9.5	2	42	868
3	25,874	1,000	98	9.8	2	43	872
3	25,874	1,000	99	9.9	2	44	895
3	25,874	1,000	100	10.0	2	45	895
3	25,874	1,000	106	10.6	3	48	1,021
3	25,874	1,000	108	10.8	3	49	1,032
3	25,874	1,000	109	10.9	3	50	1,042
3	25,874	1,000	110	11.0	3	51	1,061
3	25,874	1,000	111	11.1	3	52	1,062
3	25,874	1,000	112	11.2	3	53	1,079
3	25,874	1,000	113	11.3	3	54	1,097
3	25,874	1,000	116	11.6	3	55	1,120
3	25,874	1,000	117	11.7	3	56	1,131
3	25,874	1,000	118	11.8	3	58	1,226
3	25,874	1,000	120	12.0	3	59	1,270

TABLE 11. Or the first ten bad points

Surface	Height bound	Index bound	# points generated	% points generated	# iter	First bad point	
						Index	Height
3	25,874	1,000	137	13.7	3	69	1,496
3	25,874	1,000	151	15.1	3	82	1,741
3	25,874	1,000	164	16.4	3	95	2,110
3	25,874	1,000	177	17.7	2	107	2,458
3	25,874	1,000	187	18.7	2	118	2,753
3	25,874	1,000	207	20.7	5	134	3,039
3	25,874	1,000	220	22.0	5	146	3,391
3	25,874	1,000	233	23.3	5	160	3,928
3	25,874	1,000	243	24.3	5	174	4,686
3	25,874	1,000	255	25.5	5	184	4,865
3	25,874	1,000	268	26.8	5	197	5,257

TABLE 12. Or even the first twenty bad points

Surface	Height bound	Index bound	# points generated	% points generated	# iter	First bad point	
						Index	Height
3	25,874	1,000	301	30.1	5	226	6,309
3	25,874	1,000	325	32.5	5	248	6,811
3	25,874	1,000	347	34.7	5	269	7,255
3	25,874	1,000	367	36.7	5	290	7,873
3	25,874	1,000	388	38.8	5	314	8,592
3	25,874	1,000	409	40.9	5	338	9,134
3	25,874	1,000	434	43.4	5	359	9,673
3	51,663	2,000	536	26.8	5	359	9,673
3	124,062	5,000	734	14.7	5	359	9,673
3	251,103	10,000	985	9.9	5	359	9,673
3	505,619	20,000	1,298	6.5	7	359	9,673