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The André-Oort conjecture for products of modular curves

Emmanuel Ullmo and Andrei Yafaev

Abstract. In this paper we prove (assuming the Generalised Riemann Hypothesis) the André-Oort conjecture for products of modular curves using a combination of Galois-theoretic and ergodic-theoretic methods.

1. Introduction

The André-Oort conjecture stated below has been recently proved by Klingler, Ullmo and Yafaev (see [**UY06**] and [**KY06**]) in full generality assuming the Generalised Riemann Hypothesis.

Conjecture 1.1 (André-Oort). *Let S be a Shimura variety and let Σ be a set of special points in S. Every irreducible component of the Zariski closure of Σ is a special (or Hodge type) subvariety of S.*

For generalities on this conjecture, in particular for the notions of special points and subvarieties we refer, for example, to [**Yaf07**]. The purpose of this note is to present a proof of this conjecture in the special case where S is a product of an arbitrary number of modular curves. It is our hope that this will help in understanding the strategy used in [**UY06**] and [**KY06**], as many of the technical problems occurring in the general case do not present themselves in the case considered in this paper but all of the main ideas of the proof are conserved. The main result of this paper is the following.

Theorem 1.2. *Assume the GRH for imaginary quadratic fields. Let $n \geq 1$ be an integer and let S be a product of n modular curves. Let Σ be a set of special points in S. The irreducible components of the Zariski closure of Σ are special subvarieties.*

Note that this case of the conjecture has already been dealt with by Edixhoven [**Edi05**] but his strategy does not seem to be easily generalisable as it relies on the very particular geometric properties of the Shimura variety under consideration. We also point out that our strategy yields a proof of the Manin-Mumford conjecture as well (the "abelian counterpart" of the André-Oort conjecture). We refer to [**RU**] for details on this.

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The strategy of the proof is based on the following alternative in the geometry of Shimura varieties. Let S be a Shimura variety and let Z_n be a sequence of irreducible special subvarieties of S. Let F be some number field over which S admits a canonical model. After possibly replacing Z_n by a subsequence and assuming the GRH for CM-fields, at least one of the following cases occurs.

1. The cardinality of the sets $\{\sigma(Z_n), \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)\}$ is unbounded as $n \to \infty$ (and therefore Galois-theoretic techniques can be used).
2. The sequence of probability measures μ_n canonically associated to Z_n weakly converges to some μ_Z, the probability measure canonically associated to a special subvariety Z of S. Moreover, for every n large enough, Z_n is contained in Z.

Which of the two cases occurs depends on the geometric nature of the subvarieties Z_n.

Let us explain this in more detail in the case considered in this paper. So let S be a product of n modular curves. We assume that S is $(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})^n = \mathbb{C}^n$. Special subvarieties are products of factors which are of one of the following forms:

1. A special point (equivalently CM point) of some $\mathbb{C}^m$, $m \leq n$.
2. A modular curve $\Gamma\backslash\mathbb{H}$ (for some congruence subgroup of Γ of $\mathrm{SL}_2(\mathbb{Z})$) embedded in a product of copies of $\mathbb{C}$.

A special subvariety is called *strongly* special if it does not have any CM factors. Sequences of strongly special subvarieties are precisely those for which the second case of the alternative occurs (this is a consequence of a theorem of Clozel-Ullmo that we will recall later). The sequences of special subvarieties that do have special factors are those for which the first case of the alternative occurs.

The strategy of the proof is as follows. For a special subvariety Z, we let $c(\Omega_Z)$ be the number of CM factors, therefore $c(\Omega_Z) = 0$ means precisely that Z is strongly special. Let X be a subvariety of S containing a Zariski dense set Σ of special subvarieties. We can assume (after possibly replacing Σ by a Zariski dense subset) that $c(\Omega_Z)$ is constant as Z ranges through Σ; let's call $c(\Sigma)$ this number. If $c(\Sigma) = 0$, then X is special by the theorem of Clozel and Ullmo, otherwise the size of the Galois orbit of Z is unbounded as Z ranges through Σ. Using the explicit description of the Galois action on special points and a characterisation of special subvarieties in terms of Hecke correspondences, we show that every Z with sufficiently large Galois orbit is contained in a special subvariety Z' with $c(\Omega_{Z'}) < c(\Omega_Z)$. Thus we construct a Zariski-dense set Σ' of special subvarieties with $c(\Sigma') < c(\Sigma)$. We then reiterate the process with Σ' instead of Σ. Eventually we obtain a Zariski dense set of strongly special subvarieties.

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2. Preliminaries.

Before we state and prove our main result, we recall some definitions and prove some preliminary results which will be used in the course of the proof. Let us first recall the following definition from Edixhoven ([**Edi05**], definition 1.1).

Definition 2.1. *Let I be a finite set of cardinality r. For every i in I, let Γ_i be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and S be the product of the $X_{\Gamma_i} := \Gamma_i\backslash\mathbb{H}$ for $i \in I$. A closed irreducible subvariety Z of S is called special (of type $\Omega = \Omega_Z$) if I has a partition $\Omega = (I_1, \dots, I_t)$ such that Z is a product of subvarieties Z_i of $S_i = \prod_{j\in I_i} \Gamma_j\backslash\mathbb{H}$, each of one of the forms:*

1. *I_i is a one-element set and Z_i is a CM point.*
2. *Z_i is the image of $\mathbb{H}$ in S_i under the map sending τ in $\mathbb{H}$ to the image of $(g_s\tau)_{s\in I_i}$ in S_i, where the g_s are some elements of $\mathrm{GL}_2(\mathbb{Q})$ with positive determinant.*

Given a special subvariety Z of type Ω, we define $c(\Omega)$ to be the number of CM factors. A special subvariety Z is called strongly special if $c(\Omega) = 0$

We now prove a few lemmas that will be used in the course of the proof.

Lemma 2.2. *Let Z be a strongly special subvariety of $\mathbb{C}^n$. Then Z is defined (as an absolutely irreducible subscheme) over an abelian extension L of $\mathbb{Q}$ such that $\mathrm{Gal}(L/\mathbb{Q})$ is killed by the multiplication by 2, i.e. for every σ in $\mathrm{Gal}(L/\mathbb{Q})$, $\sigma^2 = 1$.*

Proof. This is a consequence of the explicit description of the Galois action on irreducible components of strongly special subvarieties via a reciprocity law. We refer to section 2 of [**UY06**] for details on this.

The inclusion $Z \hookrightarrow \mathbb{C}^n$ corresponds to the inclusion of Shimura data

$$(\mathrm{PGL}_2^m, \mathbb{H}^{\pm m}) \hookrightarrow (\mathrm{PGL}_2^n, \mathbb{H}^{\pm n})$$

for some $m \leq n$. This is a consequence of the explicit description of special subvarieties of $\mathbb{C}^n$ given above. Let $\rho\colon \mathrm{SL}_2^m \longrightarrow \mathrm{PGL}_2^m$ be the simply connected covering. Its kernel is killed by 2. Then the reciprocity morphism defining the Galois action on connected components is a morphism

$$r\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{PGL}_2^m(\mathbb{A}_f)/\mathrm{PGL}_2^m(\mathbb{Q})\rho(\mathrm{SL}_2^m(\mathbb{A}_f))$$

It is now clear that its image (which is isomorphic to $\mathrm{Gal}(L/\mathbb{Q})$) is killed by 2. $\square$

Using the above lemma we now prove the following.

Lemma 2.3. *Let $Z = \{x_1, \dots, x_s\} \times Z'$ be a special subvariety of $\mathbb{C}^n$ (Z' is a strongly special subvariety of $\mathbb{C}^{n-s}$ and $(x_1, \dots, x_s)$ is a special point of $\mathbb{C}^s$). Let O_{x_i} be the ring of complex multiplication of the point x_i. Let l be a prime splitting every O_{x_i}. Let $T_{\mathbf{l^2}}$ be the Hecke correspondence defined by the element of the product of r copies of $\mathrm{GL}_2(\mathbb{Q})^+$ which is $\left(\begin{smallmatrix} l^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ on the first s components and 1 elsewhere.*

There exists an element σ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that

$$\sigma(Z) \subset T_{\mathbf{l^2}} Z$$

Proof. Let K be the composite of the fields K_{x_i} of complex multiplication of the points x_i and let R be the ring $O_{x_1} \otimes \cdots \otimes O_{x_s}$. The ring R is an order in K and the prime l splits in R. Let τ be the Frobenius element in $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ for a prime ideal lying over l. The theory of complex multiplication of elliptic curves shows that $\tau^2(x_i) \subset T_{l^2}(x_i)$ where T_{l^2} is the usual Hecke correspondence given by the element

$\left(\begin{smallmatrix} l^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ of $\mathrm{GL}_2(\mathbb{Q})^+$. By the lemma above $\tau^2(Z') = Z'$. It follows that we can take $\sigma = \tau^2$. □

As in [**Edi05**], we will make use of lower bounds for Galois orbits of CM points. Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and x a special point of $\Gamma\backslash\mathbb{H}$. Let O_x be the ring of complex multiplication of x (an order in an imaginary quadratic field) and d_x be the absolute value of the discriminant of O_x. Let Z be a special subvariety of S of type Ω with $s = c(\Omega) > 0$. Let, as in the above lemma, $\{x_1, \dots, x_s\}$ be the set of CM points occurring as a CM factor of Z and let $d_Z = \max_{1\leq i\leq s} d_{x_i}$. In the following statement and in the rest of this paper, the symbol $\gg$ stands for "up to a uniform constant". We hope that this notation will cause no confusion.

Proposition 2.4. *Let $0 < \epsilon < 1/2$ be a real number. Let Z be a special subvariety of type Ω with $c(\Omega) > 0$. The following inequality holds:*

$$|\{\sigma(Z), \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\}| \gg d_Z^{\frac{1}{2}-\epsilon}$$

Proof. The cardinality of this set is at least that of the Galois orbit of the special point $(x_1, \dots, x_s)$ of a product of s modular curves. The lower bound for this Galois orbit is a consequence of the Brauer-Siegel theorem. We refer to section 5 of [**Edi05**] for details. □

To finish this section, we state the following consequence of the main theorem of [**CU05**] that will be used in a crucial way in the course of our proof.

Theorem 2.5 (Clozel-Ullmo). *Let S be a product of modular curves and let Z_n be a sequence of strongly special subvarieties. The sequence Z_n is equidistributed in the following sense. There exists a subsequence Z_{n_k} and a strongly special subvariety Z such that the sequence of probability measures μ_{n_k} canonically associated to Z_{n_k} weakly converges to μ_Z, the probability measure canonically associated to Z. Furthermore, Z contains Z_{n_k} for all k large enough.*

3. A characterization.

In this section we consider a subvariety X containing a special subvariety Z with $c(\Omega_Z) > 0$. We prove that if Z is contained in its image by some suitable Hecke correspondence, then X contains a special subvariety Z' containing Z properly. This is a key ingredient of our proof.

We will make use of the notion of degree of a subvariety V of ${\mathbb{P}^1}^r$ that we now recall. The Chow ring of ${\mathbb{P}^1}^r$ is $\mathbb{Z}[\epsilon_1, \dots, \epsilon_r]$ with $\epsilon_i^2 = 0$. Suppose that V is irreducible of codimension i. The class $[V]$ of V in the Chow group $CH^i({\mathbb{P}^1}^r)$ is

$$[V] = \sum_{|I|=i} a_I \epsilon_I$$

where ϵ_I is the product of the ϵ_i for i in I and a_I is the degree of the projection of V onto the product of copies of ${\mathbb{P}^1}^r$ indexed by the complement $I^\vee$ of I in $\{1, \dots, r\}$. We define the degree of V to be $\deg(V) = \sum_I a_I$. The variety S admits a quasi-finite morphism to $\mathbb{C}^r$. Let V' be a subvariety of S and let V be the closure of the image of V' in ${\mathbb{P}^1}^r$. The morphism from V' to V is quasi-finite. We define the degree of V' to be $\deg(V') = \deg(V)[\mathbb{C}(V') : \mathbb{C}(V)]$, where $\mathbb{C}(V')$ and $\mathbb{C}(V)$ denote the function fields of V' and V, respectively. The degree $[\mathbb{C}(V') : \mathbb{C}(V)]$ is at most the index of the product of the Γ_i in the product of the $\mathrm{SL}_2(\mathbb{Z})$.

Proposition 3.1. *Let $r \geq 3$ be an integer. Let X be an irreducible hypersurface of $\mathbb{C}^r$ such that for every $I \subset \{1, \dots, r\}$ of cardinality $r-1$, the projections $p_I \colon X \longrightarrow \mathbb{C}^{|I|}$ are dominant (in particular generically finite). Let l be a prime. Let $s \leq r$ be an integer and let T_{l^2} be the Hecke correspondence defined by the element of the product of r copies of $\mathrm{GL}_2(\mathbb{Q})^+$ which is $\left(\begin{smallmatrix} l^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ on the first s components and 1 elsewhere.*

Suppose that l is larger than $\deg(X)$ and 13. Then the variety $T_{l^2}X$ is irreducible.

Proof. Let $Y(l^2)$ be $\Gamma(l^2)\backslash\mathbb{H}$ where $\Gamma(l^2) := \ker(\mathrm{SL}_2(\mathbb{Z}) \longrightarrow \mathrm{SL}_2(\mathbb{Z}/l^2\mathbb{Z}))$. Let π be the quotient map $Y(l^2)^r \longrightarrow \mathbb{C}^r$. Let l be a prime as in the statement above. The proof of Proposition 4.2 of [**Edi05**] shows that $\pi^{-1}X$ is irreducible. As $T_{l^2}X$ is the image of $\pi^{-1}X$ by some (other) morphism, $T_{l^2}X$ is irreducible as well. □

We recall (see [**Edi05**]) that if X is an irreducible subvariety of $\mathbb{C}^r$ then a minimal projection p_I for X is given by a subset $I \subset \{1, \dots, r\}$ such that $p_I X$ is a hypersurface of $\mathbb{C}^{|I|}$ such that for all subsets I' of I with $|I'| = |I| - 1$, the projection $p_{I'}X$ is dominant. We now prove our fundamental characterisation.

Proposition 3.2. *Let X be a subvariety of $\mathbb{C}^r$ all of whose irreducible components have the same minimal projections.*

Suppose that every irreducible component X_i of X contains a special subvariety Z_i with $s = c(\Omega_{Z_i}) > 0$ independent of i and such that

$$Z_i = \{x_1, \dots, x_s\} \times Z_i'$$

where, as usual, x_is are CM points and Z_i' is strongly special.

Suppose that the first projection of X (or of one of its irreducible components) is dominant.

Suppose that there exists a prime l greater than 13 and greater than $\deg(X)$ such that

$$X \subset T_{l^2}X$$

Then X is a direct product $X = \mathcal{X}_1 \times \mathcal{X}_2$ of subvarieties of $\mathbb{C}^s$ and $\mathbb{C}^{r-s}$ respectively. Furthermore, the irreducible components of $\mathcal{X}_1$ are special. In particular, each component X_i of X contains a special subvariety Z_i' containing Z_i with $c(\Omega_{Z_i'}) < c(\Omega_{Z_i})$.

Proof. We will use the following lemma.

Lemma 3.3. *Let I be a subset of $\{1, \dots, r\}$ which is minimal for every component of X. Then either I is contained in $\{1, \dots, s\}$ or I is contained in $\{s+1, \dots, r\}$. Furthermore, if $|I| \geq 3$, then I is contained in $\{s+1, \dots, r\}$.*

Proof. Suppose that $|I| \geq 3$. We will show that in this case the intersection of I with $\{1, \dots, s\}$ is empty. Suppose it is not and write $I = I_1 \coprod I_2$ where I_1 is the intersection of I with $\{1, \dots, s\}$. Let x_2 be a point of $p_{I_2}X$. As I is minimal, the irreducible components of the subvarieties $p_I X$ and $p_I(T_l X)$ of $\mathbb{C}^{|I|}$ are hypersurfaces. The degree of $p_I X$ is at most the degree of X. Proposition 3.1 above shows that for every component Y of $p_I X$, $p_I(T_l)Y$ is irreducible, hence $p_I X$ and $p_I T_l X$ have the same number of irreducible components. The inclusion $X \subset T_l X$ implies that $p_I X$ contains a $p_I T_l$-orbit. Lemma 4.4 of [**Edi05**] shows that the orbits of the usual Hecke correspondence T_l on $\mathbb{C}$ are dense. Hence $p_I X$ contains $\mathbb{C}^{I_1} \times \{x_2\}$. It follows that components of $p_I X$ are of the form $\mathbb{C}^{I_1} \times p_{I_2} X_i$

(X_i being the components of X) which contradicts the minimality of I (note that for any i in I_1, the projection on $(I_1 - \{i\}) \cup I_2$ is not dominant).

Suppose now that $|I| = 2$. We will see that either I is contained in $\{1, \dots, s\}$ or in $\{s+1, \dots, r\}$. Indeed, suppose that I is not contained in one of the two sets. Let Z be a special subvariety contained in X as in the statement. It follows that $p_I Z$ is of the form $\{x\} \times \mathbb{C}$ where x is some special point. By minimality of I, this implies that components of $p_I X$ are of the form $\{x\} \times \mathbb{C}$. But this again contradicts the minimality of the set I.

Finally, it can occur that I is a one-element set but then $p_I(X)$ is a special point among $\{x_2, \dots, x_s\}$ and hence I is again contained in $\{1, \dots, s\}$. $\square$

By Lemma 3.5 of [**Edi05**], X is a union of components of the intersection of the $p_I^{-1} p_I X$ with I ranging through minimal sets for X. The intersection of the $p_I^{-1} p_I X$ for I contained in $\{1, \dots, s\}$ is of the form $\mathcal{X}_1' \times \mathbb{C}^{r-s}$. The intersection for the I contained in $\{s+1, \dots, r\}$ is of the form $\mathbb{C}^s \times \mathcal{X}_2'$. It follows that X is a union of components of the product $\mathcal{X}_1' \times \mathcal{X}_2'$, say $X = \mathcal{X}_1 \times \mathcal{X}_2$.

It remains to see that the components of $\mathcal{X}_1$ are special. Fix a component say $X_1 = Y_1 \times Y_2$ of X.

Let I be a minimal set for Y_1. Then I is contained in $\{1, \dots, s\}$ and the lemma above shows that either $|I| = 2$ or $|I| = 1$. By Proposition 3.6 of [**Edi05**], it suffices to show that the closure of the image of every minimal projection $p_I(Y_1)$ is special.

If $|I| = 1$, then the projection $p_I Y_1$ of Y_1 is a special point.

Suppose now that $|I| = 2$. Then (the closure of) $p_I Y_1$ is an irreducible curve in $\mathbb{C}^2$. Let $\mathcal{Y}$ be this curve. Furthermore, the degrees of the two projections of $\mathcal{Y}$ are finite (because I is minimal) and bounded above by $\deg(X)$. The proof of Proposition 4.1 of [**Edi05**] shows that $\mathcal{Y}$ is special.

As every minimal set I for Y_1 satisfies $|I| \leq 2$ and $p_I Y_1$ is special, Proposition 3.6 of [**Edi05**] shows that Y_1 is special.

Write $Z_1 = \{x_1, \dots, x_s\} \times \mathcal{Z}_1$ where $\mathcal{Z}_1$ is a strongly special subvariety of $\mathbb{C}^{r-s}$. We now take $Z_1' = Y_1 \times \mathcal{Z}_1$. $\square$

4. Proof of the main theorem.

This section is devoted to the proof of the following theorem, which is the main result of this note.

Theorem 4.1. *Assume the GRH for imaginary quadratic fields. Let X be a subvariety of a product S of r modular curves containing a Zariski dense set Σ of special subvarieties. Then the irreducible components of X are special.*

We can assume that all the subvarieties in Σ are of the same type Ω. As X contains a Zariski dense set of special points, X is defined over $\overline{\mathbb{Q}}$. We replace X by a (finite) union of its $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugates and hence assume that X is irreducible over $\mathbb{Q}$. If $c(\Omega) = 0$, then X is special by the theorem 2.5. Using Proposition 2.1 of [**EY03**], we see that we can (and do) assume that S is the product of r copies of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ (this is the fact that the level structure does not matter for the André-Oort conjecture). We can also assume that the projections of X to every factor are dominant (simply remove the factors to which X projects as just one, necessarily special, point).

Let s denote $c(\Omega)$. Renumbering the I_i and possibly replacing Σ by a Zariski dense subset allows us to assume that the cardinality of I_i is one for $i = 1, \dots, s$

and the corresponding projection is a CM point. In other words a subvariety Z in Σ can be written as

$$Z = \{x_1, \dots, x_s\} \times Z'$$

with Z' strongly special (in the product of the $\Gamma_i \backslash \mathbb{H}$ for $i > s$). We can also assume that $d_Z = d_{x_1}$ (renumbering the x_i). Our main theorem is a consequence of the following proposition, the proof of which will occupy the rest of this section.

Proposition 4.2. *Suppose that* $c(\Omega) > 0$*, then* X *contains a Zariski dense set of special subvarieties of type* Ω' *with* $c(\Omega') < c(\Omega)$*.*

Lemma 4.3. *Assume the GRH for imaginary quadratic fields. Fix an integer* $A \geq 3$*. As soon as* d_Z *is larger than some absolute constant, there exists a prime* l *which is split in every* O_{x_i} *and satisfies*

$$(\log d_Z)^A < l < (\log d_Z)^{A+1}$$

Proof. Let us quickly recall a consequence of the effective Chebotarev theorem (that assumes the GRH) in the form presented in the section 6 of [**Edi05**]. We also use [**Edi01**] appendix N. Edixhoven shows that for a given real number $x > (\log d_Z)^3$ (and bigger than some absolute constant), the number $\pi(x)$ of primes $l \leq x$ split in every O_{x_i} satisfies

$$|\pi(x) - \frac{\mathrm{Li}(x)}{n_K}| \leq \frac{\sqrt{x}}{3n_K}(\log(d_Z) + n_K \log(d_Z)) \tag{1}$$

where n_K is the degree of the composite K of the fields of complex multiplication of the x_i. Here $\mathrm{Li}(x) = \int_2^x dt/\log(t)$. Edixhoven further shows in the appendix N to [**Edi01**] that if x is larger than $(\log d_Z)^3$, then $(\log d_Z)\log(x)/3\sqrt{x} < 1/2$. Using this and the facts that $Li(x)\log(x)/x$ tends to 1 and $(\log x)^2/\sqrt{x}$ tends to 0 when x tends to infinity, we deduce that for $x \geq (\log d_Z)^3$ and larger than some absolute constant

$$\frac{x}{3n_K \log(x)} \leq \pi(x) \leq \frac{3x}{2n_K \log(x)}$$

The number of primes we are interested in is

$$\pi((\log d_Z)^{A+1}) - \pi((\log d_Z)^A)$$

Hence the number of primes l satisfying $(\log d_Z)^A < l < (\log d_Z)^{A+1}$ is at least

$$\frac{(\log d_Z)^A}{3n_K \log\log(d_Z)}(\frac{(\log d_Z)}{A+1} - \frac{9}{2A})$$

which is clearly larger than 1 provided d_Z is larger than some absolute constant. □

As Z ranges through Σ, d_Z is unbounded because the first projection of X is dominant and the projection of Σ is Zariski dense (infinite) in $\mathbb{C}$. We can now finish the proof of Theorem 4.2 and hence of 4.1 by induction.

Using equation (1), we choose a prime $l > \max(3, \deg(X))$ split in every O_{x_i} and satisfying

$$l < (\log d_Z)^3$$

If X is contained in $T_{l^2}X$, then Proposition 3.2 shows that X contains a special subvariety Z' containing Z with $c(\Omega_{Z'}) < c(\Omega_Z)$.

Suppose now that a geometrically irreducible component X' of X is not contained in $T_{l^2}X$. As both X and $T_{l^2}X$ are defined over $\mathbb{Q}$, either the intersection of X with $T_{l^2}X$ is proper or X is contained in $T_{l^2}X$. We make use of the following lemma.

Lemma 4.4. *Suppose that X is not contained in $T_{\mathfrak{l}^2}X$. We can choose a hypersurface H in $(\mathbb{P}^1_{\mathbb{Q}})^r$ such that*

(1) *X is not contained in H but $T_{\mathfrak{l}^2}(X) \subset H$.*
(2) $\deg(H) \ll \deg(X) l^{2s}$.

Proof. Let $\overline{X}$ be the closure of X in $(\mathbb{P}^1_{\mathbb{Q}})^r$. Let $[\overline{X}] = \sum_{|I|=r-\dim(X)} a_I \epsilon_I$ be the decomposition of the cycle $[\overline{X}]$ in $CH^{r-\dim(X)}({\mathbb{P}^1}^r)$. Then

$$[\overline{T_{\mathfrak{l}^2}(X)}] = \sum_{|I|=r-\dim(X)} (l^2+l)^s a_I \epsilon_I$$

We choose H to be a hypersurface such that

$$[H] = \Big(\dim(X)! \cdot \sum_{|I|=r-\dim(X)} (l^2+l)^s a_I\Big) \sum_{i=1,\dots,r} \epsilon_i$$

As usual, write $Z = \{x_1, \dots, x_s\} \times Z'$. As l splits every O_{x_i}, Lemma 2.3 implies that some Galois conjugate of Z is contained in $T_{\mathfrak{l}^2}Z \subset T_{\mathfrak{l}^2}X$. By rationality of the Hecke operator $T_{\mathfrak{l}^2}$ we find that $Z \subset X \cap T_{\mathfrak{l}^2}X$. Hence $Z \subset X \cap H$. □

Lemma 4.5. *Let Y be a $\mathbb{Q}$-component of $X \cap H$ containing Z.The projection of Y on the first factor is dominant.*

Proof. Suppose that the first projection of Y is not dominant. Then some geometrically irreducible component of Y is of the form $\{x_1\} \times Y'$. Therefore, the image of the first projection of Y contains the Galois orbit of x_1. It follows that $[Y]$ is divisible by $O(x_1)\epsilon_1$, where $O(x_1)$ denotes the cardinality of this Galois orbit. It follows that the degree of Y is at least $O(x_1)$. The fact that

$$O(x_1) \gg d_Z^{\frac{1}{2}-\epsilon}$$

contradicts the fact that the degree of H is bounded by a uniform power of $\log(d_Z)$. □

We replace $X := X_1$ by $X_2 := Y$, given by the previous lemma. The degree of X_2 is $\ll (\log d_Z)^A$ where A is some uniform integer. Using Lemma 4.3 we can find a prime l_2 split in every O_{x_i} such that

$$\deg(X_2) < l_2 \ll (\log d_Z)^{A+1}.$$

We now apply the construction just made recursively. If the inclusion did not occur at any of the previous stages, then we end up in the following situation.

(1) $\dim(X_k) = \dim(Z) + 1$
(2) $\deg X_k \ll (\log d_Z)^C$ where C is some uniform integer.
(3) $|\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot Z| \gg d_Z^{\frac{1}{2}-\epsilon}$.
(4) $Z \subset X_k$

Using effective Chebotarev, provided d_Z is large enough, we can choose l_k splitting the O_{x_i}'s such that for an absolute constant C

(1) $l^{2s} \ll (\log d_Z)^C = o(d_Z^{\frac{1}{2}-\epsilon})$ for a small $\epsilon > 0$.
(2) $l_k > \deg(X_k)$.

Then by the lemma 2.3, $Z \subset X_k \cap T_{\mathfrak{l}_k^2}X_k$. The inequalities above show that the intersection $X_k \cap T_{\mathfrak{l}_k^2}X_k$ is not proper, hence X_k is contained in $T_{\mathfrak{l}_k^2}X_k$ and by 3.2 X_k and therefore X contains a special subvariety Z' containing Z with $c(\Omega_{Z'}) < c(\Omega_{Z'})$. This finishes the proof.

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Université de Paris-Sud, Bat 425 et institut universitaire de France, 91405, Orsay Cedex France

E-mail address: `ullmo@math.u-psud.fr`

University College London, Department of Mathematics, 25 Gordon street, WC1H OAH, London, United Kingdom

E-mail address: `yafaev@math.ucl.ac.uk`