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Lectures on Modular Symbols

Yuri I. Manin

Abstract. In these lecture notes, written for the Clay Mathematics Institute Summer School "Arithmetic Geometry", Göttingen 2006, I review some classical and more recent results about modular symbols for $SL(2)$, including arithmetic motivations and applications, an iterated version of modular symbols, and relations with the "non–commutative boundary" of the modular tower for elliptic curves.

1. Introduction: arithmetic functions and Dirichlet series

1.1. Arithmetic functions. Many basic questions of number theory involve the behavior of *arithmetic functions*, i.e. sequences of integers $\{a_n \,|\, n \geq 1\}$ defined in terms of divisors of n, or numbers of solutions of a congruence modulo n, etc. After having chosen such a function, one might ask for example:

(i) Is $\{a_n \,|\, n \geq 1\}$ multiplicative, that is, does $a_{mn} = a_m a_n$ for $(m,n) = 1$?

(ii) What is the asymptotic behavior of $\sum_{n \leq N} a_n$ as $N \to \infty$?

(iii) Can one give a "formula" for a_n if initially it was introduced only by a computational prescription, such as $a_n :=$ *the number of representations of* n *as a sum of four squares*?

A very universal machinery for studying such questions consists in introducing *a generating series* for a_n depending on a complex parameter, and studying the analytic and algebraic properties of this series.

Two classes of series that are used most often are the Fourier series

$$f(z) := \sum_{n=1}^{\infty} a_n e^{2\pi i n z} \tag{1.1}$$

and the Dirichlet series

$$L_f(s) = \sum_{n=1}^{\infty} a_n n^{-s}. \tag{1.2}$$

In full generality, they must be considered as formal series; however, if a_n does not grow too fast, e.g. is bounded by a polynomial in n, then (1.1) converges in the

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upper half-plane $H := \{z \in \mathbf{C} \,|\, \mathrm{Im}\, z > 0\}$, whereas (1.2) converges in some right half plane $\mathrm{Re}\, s > D$.

1.2. Mellin transform and modularity. Some of the properties of $\{a_n\}$ are directly encoded in the generating Dirichlet series. For example, multiplicativity of $\{a_n\}$ translates into the existence of an Euler product over primes p:

$$L_f(s) = \prod_p L_{f,p}(s), \qquad L_{f,p}(s) := \sum_{n=1}^{\infty} a_{p^n} p^{-ns}. \tag{1.3}$$

Hence the Dirichlet series for the logarithmic derivative of such a function carries information about the values of a_n restricted to powers of primes. This idea leads to famous "explicit formulas" expressing partial sums of a_{p^n}'s via poles of the logarithmic derivative of $L_f(s)$ i.e. essentially zeroes of $L_f(s)$. Applied to the simplest multiplicative sequence $a_n = 1$ for all n, this formalism produces the classical relationship between primes and zeroes of Riemann's zeta.

It turns out, however, that to establish the necessary analytic properties of $L_f(s)$ such as the analytic continuation in s and a functional equation, and generally even the existence of an Euler product, one should focus first upon the Fourier series $f(z)$. The main reason for this is that interesting functions $f(z)$ more often than not possess, besides the obvious periodicity under $z \mapsto z+1$, additional symmetries, for example, a simple behavior with respect to the substitution $z \mapsto -z^{-1}$. This is the case for $f(z) = \sum_{n\geq 1} e^{2\pi i n^2 z}$ (or the more symmetric $\sum_{n\in\mathbf{Z}} e^{2\pi i n^2 z}$) corresponding to $L_f(s) = \zeta(2s)$.

The transformations $z \mapsto z + 1$ and $z \mapsto -z^{-1}$ together generate the full modular group $PSL(2, \mathbf{Z})$ of fractional linear transformations of H, and Fourier series of various *modular forms* with respect to this group and its subgroups of finite index generate a vast supply of most interesting arithmetic functions.

The basic relation between $f(z)$ and $L_f(s)$ allowing one to translate analytic properties of $f(z)$ into those of $L_f(s)$ is the integral *Mellin transform*

$$\Lambda_f(s) := \int_0^{i\infty} f(z) \left(\frac{z}{i}\right)^s \frac{dz}{z}. \tag{1.4}$$

Here the s–th power in the integrand is interpreted as the branch of the exponential function which takes real values for real s and imaginary z. Convergence at $i\infty$ is usually automatic whereas convergence at 0 is justified by a functional equation (possibly after disposing of a controlled singularity).

Whenever we can integrate termwise using (1.1) (for large $\mathrm{Re}\, s$), an easy calculation shows that

$$\Lambda_f(s) = (2\pi)^{-s}\Gamma(s)L_f(s). \tag{1.5}$$

A functional equation for $f(z)$ with respect to $z \mapsto -z^{-1}$ (or more generally, $z \mapsto -(Nz)^{-1}$ for some N) then leads formally to a functional equation of Riemann type connecting $\Lambda_f(s)$ with $\Lambda_f(1-s)$ or $\Lambda_f(D-s)$ for an appropriate D defining *the critical strip* $0 \leq \mathrm{Re}\, s \leq D$ for $L_f(s)$.

This is a very classical story, which acquired its final shape in the work of Hecke in the 1920's and 30's. More modern insights concern the role of Γ–factors as Euler factors at *arithmetic infinity*, and most important, the universality of this picture and the existence of its vast generalizations crystallized in the *Taniyama–Weil conjecture* and the so-called *Langlands program.* This involves, in particular, consideration of much more general arithmetic groups than $PSL(2)$ as modular groups.

We will not discuss this vast development in these lectures and focus upon the classical modular group and related modular symbols. For some generalizations, see [**AB90**], [**AR79**].

2. Classical modular symbols and Shimura integrals

2.1. Modular symbols as integrals. Since we are interested in Mellin transforms of the form (1.4) where $f(z)$ has an appropriate modular behavior with respect to a subgroup of $PSL(2,\mathbf{Z})$, we must keep track of similar integrals taken over $PSL(2,\mathbf{Z})$–images of the upper semi–axis as well. The latter are geodesics connecting two *cusps* in the partial compactification $\overline{H} := H \cup \mathbf{P}^1(\mathbf{Q})$.

Roughly speaking, the *classical modular symbols* are linear functionals (spanned by)

$$\{\alpha,\beta\} \; : \; f \mapsto \int_\alpha^\beta f(z) z^{s-1} dz, \qquad \alpha,\beta \in \mathbf{P}^1(\mathbf{Q})$$

on appropriate spaces of 1–forms $f(z)z^{s-1}dz$. To be more precise, we must recall the following definitions.

The group of real matrices with positive determinant $GL^+(2,\mathbf{R})$ acts on H by fractional linear transformations $z \mapsto [g]z$. Let $j(g,z) := cz+d$ where (c,d) is the lower row of g. Then we have, for any function f on H and homogeneous polynomial $P(X,Y)$ of degree $k-2$,

$$\begin{aligned} g^*[f(z)\,P(z,1)\,dz] &:= f([g]z)\,P([g]z,1)\,d([g]z) \\ &= f([g]z)\,(j(g,z))^{-k} P(az+b,cz+d)\det g\,dz \end{aligned} \tag{2.1}$$

where (a,b) is the upper row of g. From the definition it is clear that the diagonal matrices act identically so that we have in fact an action of $PGL^+(2,\mathbf{R})$.

This action induces for any integer $k \geq 2$ the weight k action of $GL^+(2,\mathbf{R})$ on functions on H. In the literature one finds two different normalizations of such an action. They differ by a determinantal twist and therefore coincide on $SL(2,\mathbf{R})$ and the modular group. For example, in [**Mer94**] and [**Man06**] the action

$$f\,|[g]_k(z) := f([g]z)\,j(g,z)^{-k}\,(\det g)^{k/2} \tag{2.2}$$

is used.

A holomorphic function $f(z)$ on H is a modular form of weight k for a group $\Gamma \subset SL(2,\mathbf{R})$ if $f|[\gamma]_k(z) = f(z)$ for all $\gamma \in \Gamma$ and $f(z)$ is finite at cusps.

Such a form is called a cusp form if it vanishes at cusps.

Let $S_k(\Gamma)$ be the space of cusp forms of weight k. Denote by $Sh_k(\Gamma)$ the space of 1–forms on the complex upper half plane H of the form $f(z)\,P(z,1)\,dz$ where $f \in S_k(\Gamma)$, and $P = P(X,Y)$ runs over homogeneous polynomials of degree $k-2$ in two variables. Thus, the space $Sh_k(\Gamma)$ is spanned by 1–forms *of cusp modular type with integral Mellin arguments in the critical strip* in the terminology of [**Man06**], Def. 2.1.1, and 3.3 below.

We will now describe the space of classical modular symbols $MS_k(\Gamma)$ as the space of *linear functionals* on $S_k(\Gamma)$ spanned by the Shimura integrals

$$f(z) \mapsto \int_\alpha^\beta f(z) z^{m-1} dz; \quad 1 \le m \le k-1; \quad \alpha, \beta \in \mathbf{P}^1(\mathbf{Q}). \tag{2.3}$$

Three descriptions of $MS_k(\Gamma)$ are known:

(i) *Combinatorial (Shimura-Eichler-Manin)*: generators and relations.

(ii) *Geometric (Shokurov)*: $MS_k(\Gamma)$ can be identified with a (part of) the middle homology of the Kuga–Sato variety $M^{(k)}$.

(iii) *Cohomological (Shimura)*: The dual space to $MS_k(\Gamma)$ can be identified with the cuspidal group cohomology $H^1(\Gamma, W_{k-2})_{cusp}$, with coefficients in the $(k-2)$-nd symmetric power of the basic representation of $SL(2)$.

We give some details below.

2.2. Combinatorial modular symbols. In this description, $MS_k(\Gamma)$ appears as an explicit subquotient of the space $W_{k-2} \otimes \overline{C}$, where W_{k-2} consists of polynomial forms $P(X,Y)$ of degree $k-2$ of two variables, and $\overline{C}$ is the space of formal linear combinations of pairs of cusps $\{\alpha, \beta\} \in \mathbf{P}^1(\mathbf{Q})$. Coefficients of these linear combinations can be $\mathbf{Q}$, $\mathbf{R}$ or $\mathbf{C}$, as in the theory of Hodge structures.

Each element of the form $P \otimes \{\alpha, \beta\}$ produces a linear functional

$$f \mapsto \int_\beta^\alpha f(z)\, P(z,1) dz.$$

This is extended to all of $W_{k-2} \otimes \overline{C}$ by linearity.

Denote by C the quotient of $\overline{C}$ by the subspace generated by sums $\{\alpha, \beta\} + \{\beta, \gamma\} + \{\gamma, \alpha\}$. Since $\int_\beta^\alpha + \int_\gamma^\beta + \int_\alpha^\gamma = 0$, our linear functional (Shimura integral) descends to $W_{k-2} \otimes C$. We will still denote by $P \otimes \{\alpha, \beta\}$ the class of this element in C.

The group $GL^+(2, \mathbf{Q})$ acts from the left on W_{k-2} by (notation as in (2.1))

$$(gP)(X,Y) := P(bX - dY, -cX + aY),$$

and on C by $g\{\alpha, \beta\} := \{g\alpha, g\beta\}$. Hence it acts on the tensor product. A change of variable formula then shows that the Shimura integral restricted to $S_k(\Gamma)$ vanishes on the subspace of $W_{k-2} \otimes C$ spanned by $P \otimes \{\alpha, \beta\} - gP \otimes \{g\alpha, g\beta\}$ for all $P \in W_{k-2}$, $g \in \Gamma$.

Denote by $MS_k(\Gamma)$ the quotient of $W_{k-2} \otimes C$ by the latter subspace.

The subspace of cuspidal modular symbols $MS_k(\Gamma)_{cusp}$ is defined by the following construction. Consider the space B freely spanned by $\mathbf{P}^1(\mathbf{Q})$. Define the space $B_k(\Gamma)$ as the quotient of $W_{k-2} \otimes B$ by the subspace generated by $P \otimes \{\alpha\} - gP \otimes \{g\alpha\}$ for all $g \in \Gamma$. There is a well-defined boundary map $MS_k(\Gamma) \to B_k(\Gamma)$ induced by $P \otimes \{\alpha, \beta\} \mapsto P \otimes \{\alpha\} - P \otimes \{\beta\}$. Its kernel is denoted $MS_k(\Gamma)_{cusp}$.

By construction, any (real) modular symbol in $MS_k(\Gamma)_{cusp}$ defines a $\mathbf{C}$–valued functional $\int$ on $S_k(\Gamma)$ and in fact even on $S_k(\Gamma) \oplus \overline{S}_k(\Gamma)$.

The first result of the theory is:

Theorem (Shimura). *$\int$ is an isomorphism of $MS_k(\Gamma)_{cusp}$ with the dual space of $S_k(\Gamma) \oplus \overline{S}_k(\Gamma)$.*

2.3. Geometric modular symbols. Let $\Gamma^{(k)}$ be the semidirect product $\Gamma \ltimes (\mathbf{Z}^{k-2} \times \mathbf{Z}^{k-2})$ acting on $H \times \mathbf{C}^{k-2}$ via

$$(\gamma;\, n, m)\,(z, \zeta) := ([\gamma]z;\, j(\gamma, z)^{-1}(\zeta + zn + m))$$

where $n = (n_1, \dots, n_{k-2})$, $m = (m_1, \dots, m_{k-2})$, $\zeta = (\zeta_1, \dots, \zeta_{k-2})$, and $nz = (n_1 z, \dots, n_{k-2} z)$.

If $f(z)$ is a Γ–invariant cusp form of weight k, then

$$f(z)dz \wedge d\zeta_1 \wedge \dots \wedge d\zeta_{k-2}$$

is a $\Gamma^{(k)}$–invariant holomorphic volume form on $H \times \mathbf{C}^{k-2}$. Hence one can push it down to a Zariski open smooth subset of the quotient $\Gamma^{(k)} \setminus (H \times \mathbf{C}^{k-2})$. An appropriate smooth compactification $M^{(k)}$ of this subset is called *a Kuga–Sato variety,* cf. [**Sho76**],[**Sho80b**],[**Sho80a**].

Denote by ω_f the image of this form on $M^{(k)}$. Notice that it depends only on f, not on any Mellin argument. The latter can be accommodated in the structure of (relative) cycles in $M^{(k)}$, so that integrating ω_f over such cycles we obtain the respective Shimura integrals.

Concretely, let $\alpha, \beta \in \mathbf{P}^1(\mathbf{Q})$ be two cusps in $\overline{H}$ and let p be a geodesic joining α to β. Fix (n_i) and (m_i) as above. Construct a cubic singular cell $p \times (0,1)^{k-2} \to H \times \mathbf{C}^{k-2}$: $(z, (t_i)) \mapsto (z, (t_i(zn_i + m_i)))$. Take the S_{k-2}–symmetrization of this cell and push down the result to the Kuga–Sato variety. We will get a relative (modulo fibers of $M^{(k)}$ over cusps) cycle whose homology class is Shokurov's higher modular symbol $\{\, \alpha, \beta;\, n, m\}_\Gamma$. One easily sees that

$$\int_\alpha^\beta f(z) \prod_{i=1}^{k-2} (n_i z + m_i)\, dz = \int_{\{\, \alpha, \beta;\, n, m\}_\Gamma} \omega_f\,.$$

The singular cube $(0,1)^{k-2}$ may also be replaced by an evident singular simplex.

Theorem (Shokurov). *(i) The map $f \mapsto \omega_f$ is an isomorphism $S_k(\Gamma) \to H^0(M^{(k)}, \Omega^{k-1}_{M^{(k)}})$.*

(ii) The homology subspace spanned by Shokurov modular symbols with vanishing boundary is canonically isomorphic to the space of cuspidal combinatorial modular symbols.

2.4. Cohomological modular symbols. In this description, the space dual to $MS_k(\Gamma)$ is identified with the group cohomology $H^1(\Gamma, W_{k-2})$.

A bridge between the geometric and the cohomological descriptions is furnished by the identification of $H^1(\Gamma, W_{k-2})_{cusp}$ with the cohomology of a local system on $M_{1,1}$, namely $H^1_!(M_{1,1}, \mathrm{Sym}^{k-2} R^1\pi_*\mathbf{Q})$.

2.5. Some arithmetic applications. The formalism sketched above allows one to get some quite precise information about two classes of number–theoretic objects: *coefficients* of modular forms and *their periods*, which are essentially values of their Mellin transforms at integer points of the critical strip. For illustration, we give two examples taken from [**Man72**] and [**Man73**].

Example 1. Let

$$\Phi(z) := e^{2\pi iz}\prod_{n=1}^{\infty}(1-e^{2\pi niz})^{24} = \sum_{n=1}^{\infty}\tau(n)e^{2\pi niz}.$$

The coefficients $\tau(n)$ form a multiplicative sequence. This follows from the fact that $\Phi(z)$ is the (essentially unique) cusp form of weight 12 with respect to the full modular group; hence in particular it is an eigenform for all Hecke operators, which ensures multiplicativity.

The formalism of modular symbols leads to an expression for $\tau(n)$ through representations of n by an indefinite quadratic form. Namely, we have

$$\tau(n) = \sum_{d/n} d^{11} + \sum_{n=\Delta\Delta'+\delta\delta'} \frac{691}{18}(\Delta^8\delta^2-\Delta^2\delta^8) + \frac{691}{6}(\Delta^6\delta^4-\Delta^4\delta^6). \qquad (2.4)$$

The second summation is taken over the following set of solutions: we require that $\Delta > \delta > 0$ and either $\Delta' > \delta' > 0$, or $\Delta/n, \Delta' = n/\Delta, \delta' = 0, 0 < \delta/\Delta \le 1/2$.

Periods of $\Phi(z)$ are Shimura integrals

$$r_k(\Phi) := \int_0^{i\infty}\Phi(z)z^k dz, \qquad 0 \le k \le 10-w$$

that is, via Mellin transform,

$$r_k(\Phi) = \frac{k!i^{k+1}}{(2\pi)^{k+1}}L_\Phi(k+1).$$

The invariance of $\Phi(z)(dz)^6$ with respect to $z \mapsto -z^{-1}$ shows that

$$r_k(\Phi) = r_k(\Phi)(-1)^{k+1}r_{10-k}(\Phi).$$

Finally, the formalism of modular symbols allows one to establish that the **Q**–space spanned by periods is at most two–dimensional. More precisely,

$$(r_0 : r_2 : r_4) = (1 : -\frac{691}{2^2\cdot 3^4\cdot 5} : \frac{691}{2^3\cdot 3^2\cdot 5\cdot 7}),\ (r_1 : r_3 : r_5) = (1 : -\frac{5^2}{2^4\cdot 3} : \frac{5}{2^2\cdot 3}).$$

Example 2: a non–commutative reciprocity law. Here we start with a cusp form of weight two

$$F(z) := e^{2\pi iz}\prod_{n=1}^{\infty}(1-e^{2\pi niz})^2(1-e^{22\pi niz})^2 = \sum_{n=1}^{\infty}\lambda_n e^{2\pi niz}$$

with respect to the subgroup $\Gamma_0(11)$ of Γ.

The Mellin transform of this form can be identified with the Weil zeta function of the elliptic modular curve $\Gamma_0(11)\setminus\overline{H}$ defined over **Q**. From this it follows that for any prime $p \ne 2, 11$, we can characterize $1-\lambda_p+p$ as the number of solutions of the congruence

$$y^2 + y \equiv x^3 - x^2 - 10x - 20 \bmod p \qquad (2.5)$$

(including the infinite solution).

On the other hand, the formalism of modular symbols allows one to write for this number an expression having the same structure as (2.4):

$$1-\lambda_p+p=\sum_{p=\Delta\Delta'+\delta\delta'} y_{11}(\Delta,\delta). \tag{2.6}$$

This time, however, $y_{11}(\Delta,\delta)$ is not a polynomial: it depends only on $(\Delta:\delta) \bmod 11$: for the values of the latter $0,\infty,\pm 1,\pm 2,\pm 3,\pm 4,\pm 5 \bmod 11$, the values of y_{11} are respectively $2,-2,0,10,5,-5,-10$.

Thus, we have connected solutions modulo p of the equation (2.5) "depending on 11" as its conductor with solutions modulo 11 of the equation $p=\Delta\Delta'+\delta\delta'$ depending on p. This justifies the name "non–commutative reciprocity law" suggested for (2.6) and its generalizations in [**Man72**].

Such formulas can be used to make more explicit the exact arithmetic content of special cases of the very general and therefore somewhat abstruse Langlands formalism.

Proofs of formulas for coefficients such as (2.4), (2.6) consist of two steps. For simplicity, we will illustrate this for the case of weight two cusp form $f(z)$ which is an eigenform with respect to a Hecke operator T_n so that $T_nf=a_nf$. We integrate this identity, say, from 0 to $i\infty$ and get

$$\int_0^{\infty} T_nf dz = a_n\int_0^{i\infty} f dz.$$

Now, use the explicit definition of the Hecke operator T_n on the left hand side and make a change of variables. We will get a sum of modular symbols. Using a continued fraction trick and a lemma initially proved by Heilbronn, we finally reduce the left hand side to a sum over solutions of $n=\Delta\Delta'+\delta\delta'$.

2.6. Relations with noncommutative geometry and a real analog of p–adic integration. The role of the upper half-plane in our constructions is of course explained by the fact that it parametrizes elliptic curves: complex tori $\mathbf{C}/\langle 1,\tau\rangle$, $\tau\in H$. The action of the modular group extends to this family, and the respective quotient is a non–complete algebraic variety. The cusps $\tau\in\mathbf{P}^1(\mathbf{Q})$ can be added to compactify this quotient by degenerate elliptic curves. However, for irrational values $\theta\in\mathbf{R}\setminus\mathbf{Q}$, the quotient $\mathbf{C}/\langle 1,\theta\rangle=\mathbf{C}^*/(e^{2\pi i\theta})$ is a "bad" topological group, and the common wisdom is that it is best represented by a non–commutative space, (a version of) the quantum torus T_θ.

Tori T_θ are parametrized by $\theta\in\mathbf{R}$. However, if one considers only tori modulo Morita equivalence, then they are parametrized by $PGL\,(2,\mathbf{Z})\setminus\mathbf{P}^1(\mathbf{R})$. Set–theoretically, $PGL\,(2,\mathbf{Z})\setminus\mathbf{P}^1(\mathbf{R})=$ the set of equivalence classes of $\alpha\in[0,1)$ modulo the relation

$$\alpha\equiv\beta \;\Leftrightarrow\; \exists\, n_0,n_1\;\forall n>0,\quad k_{n+n_0}(\alpha)=k_{n+n_1}(\beta).$$

Here $k_n(\alpha)$ are successive components of the continued fraction of α.

Thus, we can imagine an "invisible boundary" of the modular tower supporting a family of non–commutative spaces, the phantom of the classical modular family.

This viewpoint was discussed in [**MM02**], see also [**Mar05**], and in particular the Gauss problem on the distribution of continued fractions and its generalizations were treated as a measure theory on the "non–commutative modular curves".

We will describe here one result of this study, which produces an "∞–adic analogue" of the theory of p–adic integration used to construct p–adic Mellin transforms of cusp forms in [**Man73**].

Fix a prime number $N > 0$ and put $G_0 = \Gamma_0(N)$. We will assume that the genus of $X_{G_0} = X_0(N)$ is ≥ 1. Consider a $\Gamma_0(N)$–invariant differential $\omega = f(z)dz$ on H such that $f(z)$ is a cusp eigenform of weight two for all Hecke operators and denote by $L_f^{(N)}(s)$ (resp. $\zeta^{(N)}(s)$) its Mellin transform (resp. Riemann's zeta) with omitted Euler N–factor. More precisely, the coefficients of $L_f^{(N)}(s)$ are Hecke eigenvalues of f.

For $\alpha \in (0,1)$, denote by $p_n(\alpha)/q_n(\alpha)$ the n–th convergent of α.

Theorem. *We have for* $\mathrm{Re}\, t > 0$*:*

$$\int_0^1 d\alpha \sum_{n=0}^{\infty} \frac{q_{n+1}(\alpha) + q_n(\alpha)}{q_{n+1}(\alpha)^{1+t}} \int_0^{q_n(\alpha)/q_{n+1}(\alpha)} f(z)\, dz =$$

$$\left[\frac{\zeta(1+t)}{\zeta(2+t)} - \frac{L_f^{(N)}(2+t)}{\zeta^{(N)}(2+t)^2} \right] \int_0^{i\infty} f(z)\, dz. \tag{2.7}$$

If $\int_0^{i\infty} f(z)\, dz \neq 0$, we can read (2.7) as an expression for $L_f^{(N)}(s)$ which has striking structural similarities to the p–adic Mellin integral. In particular, both formulas involve a construction of a measure out of modular symbols, on $(0,1)$ and on $\mathbf{Z}_p^*$ respectively.

The proof of (2.7) given in [**MM02**] combines an old lemma by P. Lévy with the continued fractions trick alluded to above.

The Theorem above does not involve directly the non–commutative geometry of the invisible boundary. However, it was shown in [**MM02**], Sec. 4, and [**Mar05**], Sec. 6 of Ch. 4, that modular symbols themselves can be identified with specific elements in the K–theory of this space, giving additional weight to the geometric intuition behind this picture.

3. Iterated modular symbols

3.1. Multiple zeta values and iterated integrals. The theory of iterated modular symbols (cf. [**Man06**], [**Man05**]) is a simultaneous generalization of two constructions–of classical modular symbols and of multiple zeta values–and is an elaboration of a special case of Chen's iterated integrals theory ([**Che77**]) in a holomorphic setting.

Multiple zeta values are the numbers given by the k–multiple Dirichlet series

$$\zeta(m_1, \dots, m_k) = \sum_{0<n_1<\dots<n_k} \frac{1}{n_1^{m_1} \cdots n_k^{m_k}} \tag{3.1}$$

which converge for all integer $m_i \geq 1$ and $m_k > 1$, or equivalently by the m–multiple iterated integrals, $m = m_1 + \cdots + m_k$,

$$\zeta(m_1, \ldots, m_k) = \int_0^1 \frac{dz_1}{z_1} \int_0^{z_1} \frac{dz_2}{z_2} \int_0^{z_2} \cdots \int_0^{z_{m_{k-1}}} \frac{dz_{m_k}}{1 - z_{m_k}} \cdots \tag{3.2}$$

where the sequence of differential forms in the iterated integral consists of consecutive subsequences of the form $\frac{dz}{z}, \ldots, \frac{dz}{z}, \frac{dz}{1-z}$ of lengths $m_k, m_{k-1}, \ldots, m_1$.

Easy combinatorial considerations allow one to express in two different ways products $\zeta(l_1, \ldots, l_j) \cdot \zeta(m_1, \ldots, m_k)$ as linear combinations of multiple zeta values.

If one uses for this the integral representation (3.2), one gets a sum over shuffles which enumerate the simplices of highest dimension occurring in the natural simplicial decomposition of the product of two integration simplices.

If one uses instead (3.1), one gets sums over shuffles with repetitions which enumerate some simplices of lower dimension as well.

These relations and their consequences are called double shuffle relations. Both types of relations can be succinctly written down in terms of formal series on free noncommuting generators. One can include in these relations regularized multiple zeta values for arguments where the convergence of (3.1), (3.2) fails. A clear and systematic exposition of these results can be found in [**Del01**] and [**Rac00**], [**Rac02**].

In fact, the formal generating series for (regularized) iterated integrals (3.2) appeared in the famous Drinfeld paper [**Dri90**], essentially as *the Drinfeld associator*, and more relations for multiple zeta values were implicitly deduced there. The question about interdependence of (double) shuffle and associator relations does not seem to be settled at the moment of writing this: cf. [**Rac04**]. The problem of completeness of these systems of relations is equivalent to some difficult transcendence questions.

Multiple zeta values are interesting, because they and their generalizations appear in many different contexts involving mixed Tate motives ([**DG05**], [**Ter02**]), deformation quantization ([**Kon99**]), knot invariants, etc.

In order to make contact with modular symbols, notice first that the differentials $\frac{dz}{z}, \frac{dz}{1-z}$ span the space of meromorphic differential forms with no more than logarithmic singularities at points $\{0, 1, \infty\}$ of $\mathbf{P}^1(\mathbf{C})$. We can identify

$$(\mathbf{P}^1(\mathbf{C}), \{0, 1, \infty\}) \cong \Gamma_0(4) \setminus (\overline{H}, cusps).$$

Then $\frac{dz}{z}, \frac{dz}{1-z}$ lift to Eisenstein series of weight two for $\Gamma_0(4) \subset SL(2, \mathbf{Z})$.

In the general theory sketched below, $\Gamma_0(4)$ is replaced by an arbitrary (congruence) subgroup Γ of $SL(2, \mathbf{Z})$, Eisenstein series of weight two are replaced by (cusp form + Eisenstein series) with respect to Γ, multiplied by $z^{s-1}dz$ for appropriate s. (We mostly focus on cusp forms; in the presence of logarithmic singularities, the necessary regularization procedure is described for weight two in Sec. 3.6.)

Finally, ordinary integrals along geodesics connecting two cusps are replaced by iterated integrals.

3.2. Formalism of iterated integrals. We will work on a Riemann surface, and study general iterated integrals of holomorphic 1–forms. We will show that if one replaces a simple integral not by an individual iterated integral but by a generating series of all such integrals, then the usual properties like additivity and variable change formula reappear in a multiplicative/noncommutative version.

Let X be a connected complex Riemann surface, and $\omega_V := (\omega_v \,|\, v \in V)$ a family of holomorphic 1–forms indexed by a finite set V. Denote by $A_V := (A_v \,|\, v \in V)$ free associative formal variables, commuting with complex numbers, functions, and differentials on X, and put

$$\Omega := \sum_{v\in V} A_v \omega_v .$$

Consider the total iterated integral of Ω along a piecewise smooth path $\gamma : [0,1] \to U \subset X$:

$$J_\gamma(\Omega) := 1 + \sum_{n=1}^{\infty} \int_0^1 \gamma^*(\Omega)(t_1) \int_0^{t_1} \gamma^*(\Omega)(t_2) \cdots \int_0^{t_{n-1}} \gamma^*(\Omega)(t_n) \in \mathbf{C}\langle\langle A_V \rangle\rangle$$

taken over the simplex $0 < t_n < \cdots < t_1 < 1$. If γ, γ' with the same ends are homotopic then $J_\gamma(\Omega) = J_{\gamma'}(\Omega)$. Fixing implicitly such a homotopy class, we can use another notation: $z_i = \gamma(t_i) \in X$, $a = \gamma(0)$, $z = \gamma(1)$,

$$J_a^z(\Omega) := 1 + \sum_{n=1}^{\infty} \int_a^z \Omega(z_1) \int_a^{z_1} \Omega(z_2) \cdots \int_a^{z_{n-1}} \Omega(z_n) \, .$$

If $U \subset X$ is connected and simply connected, this is an unambiguously defined element of $\mathcal{O}_X(U)\langle\langle A_V \rangle\rangle$. Otherwise it is a multivalued function of z in this domain.

Proposition. *(i) $J_a^z(\Omega)$ as a function of z satisfies the equation*

$$dJ_a^z(\Omega) = \Omega(z)\, J_a^z(\Omega).$$

In other words, $J_a^z(\Omega)$ is a horizontal (multi)section of the flat connection $\nabla_\Omega := d - l_\Omega$ on $\mathcal{O}_X\langle\langle A_V \rangle\rangle$, where l_Ω is the operator of left multiplication by Ω.

(ii) If U is a simply connected neighborhood of a, $J_a^z(\Omega)$ is the only horizontal section with initial condition $J_a^a = 1$. Any other horizontal section K^z can be uniquely written in the form $CJ_a^z(\Omega)$, $C \in \mathbf{C}\langle\langle A_V \rangle\rangle$. In particular, for any $b \in U$,

$$J_b^z(\Omega) = J_a^z(\Omega) J_b^a(\Omega).$$

Corollary. *Let γ be a closed oriented contractible contour in U, $a_1, \dots, a_n$ points along this contour (cyclically) ordered compatibly with orientation. Then*

$$J_{a_2}^{a_1}(\Omega) J_{a_3}^{a_2}(\Omega) \cdots J_{a_n}^{a_{n-1}}(\Omega) J_{a_1}^{a_n}(\Omega) = 1. \tag{3.3}$$

Formula (3.3) is the multiplicative version of the additivity of simple integrals with respect to the join of integration paths.

Proposition. *Consider the comultiplication*

$$\Delta : \mathbf{C}\langle\langle A_V \rangle\rangle \to \mathbf{C}\langle\langle A_V \rangle\rangle \widehat{\otimes}_{\mathbf{C}} \mathbf{C}\langle\langle A_V \rangle\rangle, \ \Delta(A_v) = A_v \otimes 1 + 1 \otimes A_v$$

and extend it to the series with coefficients $\mathbf{C}(X)$ and $\Omega^1(X)$. Then

$$\Delta \left(J_a^z(\omega_V) \right) = J_a^z(\omega_V) \widehat{\otimes}_{\mathcal{O}_X} J_a^z(\omega_V) \, . \tag{3.4}$$

Claim 1. *The identity (3.4) encodes all shuffle relations between the iterated integrals of the forms* ω_v.

Claim 2. *The identity (3.4) is equivalent to the fact that* $\log J_a^z(\omega_V)$ *can be expressed as a series in commutators (of arbitrary length) of the variables* A_v.

Formula (3.4) expresses the group–like property of $J_a^z(\Omega)$. It is a multiplicative version of the additivity of a simple integral as a functional of the integration form.

Functorality. Let $g : X \to X$ be an automorphism such that g^* maps into itself the linear space spanned by ω_v: $g^*(\omega_v) = \sum_u g_{vu}\omega_u$. Define $g_*(A_u) = \sum_v A_v g_{vu}$. Then we have

$$J_{ga}^{gz}(\omega_V) = g_*(J_a^z(\omega_V))\,. \tag{3.5}$$

Formula (3.5) is a multiplicative version of the variable change formula.

3.3. Iterated integrals on the upper half–plane and total Mellin transform. A 1–form ω on H will be called a form of modular type if it can be represented as $f(z)z^{s-1}dz$, where s is a complex number and $f(z)$ is a modular form of some weight with respect to a finite index subgroup Γ of the modular group $SL(2,\mathbf{Z})$.

The modular form $f(z)$ is then well defined and called the associated modular form (to ω), and the number s is called the Mellin argument of ω.

ω is called a form of cusp modular type if the associated $f(z)$ is a cusp form.

Let $f_1,\dots,f_k$ be a finite sequence of cusp forms with respect to Γ, $\omega_j(z) := f_j(z)\,z^{s_j-1}dz$. The iterated Mellin transform of (f_j) is

$$M(f_1,\dots,f_k;s_1,\dots,s_k) := I_{i\infty}^0(\omega_1,\dots,\omega_k) =$$

$$\int_{i\infty}^0 \omega_1(z_1)\int_{i\infty}^{z_1}\omega_2(z_2)\cdots\int_{i\infty}^{z_{n-1}}\omega_n(z_n).$$

Let $f_V = (f_v\,|\,v\in V)$ be a finite family of cusp forms with respect to Γ, $s_V = (s_v\,|\,v\in V)$ a finite family of complex numbers, $\omega_V = (\omega_v)$, where $\omega_v(z) := f_v(z)\,z^{s_v-1}dz$. The total Mellin transform of f_V is

$$TM(f_V;s_V) := J_{i\infty}^0(\omega_V) =$$

$$1+\sum_{n=1}^{\infty}\sum_{(v_1,\dots,v_n)\in V^n} A_{v_1}\dots A_{v_n}\,M(f_{v_1},\dots,f_{v_n};s_{v_1},\dots,s_{v_n}).$$

Theorem. *Assume that the space spanned by* $f_v(z)$ *is stable with respect to* $g_N : z\mapsto -1/Nz$. *Let* k_v *be the weight of* $f_v(z)$*, and* $k_V = (k_v)$. *Then*

$$TM(f_V;s_V) = g_{N*}(TM(f_V;k_V-s_V))^{-1}$$

for an appropriate linear transformation g_{N*} *of the formal variables* A_v.

3.4. Iterated Shimura integrals and non–commutative cohomology. Let G be a group, N a group with left action of G by group automorphisms: $(g,n) \mapsto gn$. Cocycles with coefficients in N are defined as $Z^1(G,N) := \{\, u : G \to N \,|\, u(g_1 g_2) = u(g_1)\, g_1 u(g_2) \,\}$. Two cocycles are cohomologous, $u' \sim u$, iff for some $n \in N$ and all $g \in G$, we have $u'(g) = n\, u(g)\, (gn)^{-1}$. The cohomology set is $H^1(G,N) := Z^1(G,N)/(\sim)$. It is endowed with a marked point: the class of trivial cocycles $u(g) = n^{-1} \cdot gn$.

We will apply this formalism to iterated Shimura integrals. The role of G will be played by a group $G = P\Gamma \subset PSL(2,\mathbf{Z})$ where $\Gamma \subset SL(2,\mathbf{Z})$.

To define coefficients, choose as above a family of Shimura differentials $\omega_v = f_v(z) z^{m_v-1} dz$, where f_v form a basis of $\oplus_i S(k_i,\Gamma)$, and for a fixed weight, m_v runs over all critical integers for this weight. The forms ω_v span a $P\Gamma$–invariant space. Put $\Omega := \sum_{v\in V} A_v \omega_v$. The role of N will be played by $\Pi :=$ the group of group–like elements of $(1 + \sum_{v\in V} A_v \mathbf{C}\langle\langle A_v \rangle\rangle)^*$. The left action of $P\Gamma$ on Π is the functorality action g_*.

Theorem. *(i) For any $a \in \overline{H}$, the map $P\Gamma \to \Pi : \gamma \mapsto J^a_{\gamma a}(\Omega)$ is a noncommutative 1–cocycle ζ_a in $Z^1(P\Gamma,\Pi)$.*

(ii) The cohomology class of ζ_a in $H^1(P\Gamma,\Pi)$ does not depend on the choice of a and is called the noncommutative modular symbol.

(iii) This cohomology class belongs to the cuspidal subset $H^1(P\Gamma,\Pi)_{cusp}$ consisting of those cohomology classes whose restriction on all stabilizers of Γ–cusps is trivial.

Using the non–commutative Shapiro Lemma, we can reduce the general case to that of $PSL(2,\mathbf{Z})$.

Shapiro Lemma. *Let $G \subset H$ be a subgroup, N a left G–group, $N_H :=$ $\mathrm{Map}_G\,(N,H)$ with pointwise multiplication and left action of G, $(g_*\phi)(h) := \phi(hg)$. There is a canonical isomorphism of pointed sets:*

$$H^1(G,N) = H^1(H,N_H).$$

In the notation as above, we apply it to the case

$$G := P\Gamma, \quad H := PSL(2,\mathbf{Z}), \quad N := \Pi, \quad \Pi^0 := N_H.$$

It is well known that $H = PSL(2,\mathbf{Z})$ is a free product of two subgroups $\mathbf{Z}_2$ and $\mathbf{Z}_3$ generated respectively by

$$\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

Theorem. *(i) An iterated Shimura cocycle restricted to (σ,τ) belongs to the set*

$$\{\, (X,Y) \in \Pi^0 \times \Pi^0 \,|\, X \cdot \sigma_* X = 1,\, Y \cdot \tau_* Y \cdot \tau_*^2 Y = 1 \,\}.$$

(ii) The cohomology relation between cocycles translates as

$$(X,Y) \sim (m^{-1} X \sigma_*(m), m^{-1} Y \tau_*(m)).$$

(iii) The cuspidal part of the cohomology is generated by the pairs

$$\{\, (X,Y) \,|\, \exists Z,\, X \cdot \sigma_* Y = Z^{-1} (\sigma\tau)_* Z \,\}.$$

3.5. Iterated Shimura integrals as multiple Dirichlet series. Start with the family of 1–forms on H:

$$\omega_v(z) = \sum_{n=1}^{\infty} c_{v,n} e^{2\pi i n z} z^{m_v - 1} dz, \quad c_{v,n} \in \mathbf{C}, \quad m_v \in \mathbf{Z},\, m_v \geq 1;\ c_{v,n} = O(n^C).$$

Put

$$L(z; \omega_{v_k}, \dots, \omega_{v_1}; j_k, \dots, j_1) :=$$

$$(2\pi i z)^{j_k} \sum_{n_1,\dots,n_k \geq 1} \frac{c_{v_1,n_1} \cdots c_{v_k,n_k} e^{2\pi i (n_1 + \cdots + n_k) z}}{n_1^{m_{v_1} + j_0 - j_1} (n_1 + n_2)^{m_{v_2} + j_1 - j_2} \cdots (n_1 + \cdots + n_k)^{m_{v_k} + j_{k-1} - j_k}}\,.$$

Exponentials ensure absolute convergence for any z with $\operatorname{Im} z > 0$. Formal substitution $z = 0$ may lead to divergence.

Theorem. *For any $k \geq 1$, $(v_1, \dots, v_k) \in V^k$, and $\operatorname{Im} z > 0$ we have*

$$(2\pi i)^{m_{v_1} + \cdots + m_{v_k}} I^z_{i\infty}(\omega_{v_k}, \dots, \omega_{v_1}) =$$

$$(-1)^{\sum_{i=1}^k (m_{v_i} - 1)} \sum_{j_1 = 0}^{m_{v_1} - 1} \sum_{j_2 = 0}^{m_{v_2} - 1 + j_1} \cdots \sum_{j_k = 0}^{m_{v_k} - 1 + j_{k-1}} (-1)^{j_k} \times$$

$$\frac{(m_{v_1} - 1)!(m_{v_2} - 1 + j_1)! \cdots (m_{v_k} - 1 + j_{k-1})!}{j_1! j_2! \cdots j_k!} L(z; \omega_{v_k}, \dots, \omega_{v_1}; j_k, \dots, j_1)\,.$$

Proposition. *Assume that ω_V as above is a basis of a space of 1-forms invariant with respect to g_N. Then*

$$J^0_{i\infty}(\omega_V) = (g_{N*}(J^{\frac{i}{\sqrt{N}}}_{i\infty}(\omega_V)))^{-1} J^{\frac{i}{\sqrt{N}}}_{i\infty}(\omega_V)\,. \tag{3.6}$$

Replacing the coefficients of the formal series in the r.h.s of (3.6) by their (convergent) representations via multiple Dirichlet series with exponents we get such representations for $I^0_{i\infty}(\omega_{v_k}, \dots, \omega_{v_1})$ and avoid divergences at $z = 0$.

The multiple Dirichlet series generated by Shimura integrals as above do not form, however, a closed system with respect to multiplication, so that we cannot deduce an analog of shuffle relations with repetitions valid for multiple zeta values. If we complete the family of such series using a combinatorial trick described in [**Man06**], then representation of such series as iterated integrals will involve more general 1–forms than we have been considering up to now. This subject deserves a further study.

3.6. Differentials with logarithmic singularities at the endpoints of integration. We will now assume, as in the initial Drinfeld setting, that the integration limits of the iterated integral are logarithmic singularities of the form Ω. Generally, they diverge and must be regularized. The dependence on the regularization can be described as a version of Deligne's choice of the "base point at infinity".

Let a = a fixed point of the Riemann surface, z a variable point. Put $r_{v,a} := \operatorname{res}_a \omega_v$, $\quad R_a := \operatorname{res}_a \Omega = \sum_v r_{v,a} A_v$. Denote by t_a a local parameter at a, and by $\log t_a$ a local branch of the logarithm real on $t_a \in \mathbf{R}_+$. Finally, put $t_a^{R_a} := e^{R_a \log t_a}$.

Definition. *A local solution to* $dJ^z = \Omega(z)J^z$ *is called normalized at* a *(with respect to a choice of* t_a*) if it is of the form* $J = K \cdot t_a^{R_a}$*, where* K *is a holomorphic section in a neighborhood of* a *and* $K(a) = 1$*.*

Claim. *(i) The normalized solution exists and is unique.*

(ii) It depends only on the tangent vector $\partial/\partial t_a\,|_a$*.*

(iii) If $J'_a = K'(t'_a)^{R_a}$ *is normalized with respect to* t'_a*, and* $\tau_a := dt'_a/dt_a\,|_a$*, then* $J'_a = J_a \cdot \tau_a^{R_a}$*.*

Now, having chosen $(a, t_a), (b, t_b)$, a 1–form $\Omega = \sum A_v \omega_v$ with at most logarithmic singularities at a, b, and a (homotopy class of) path(s) from a to b avoiding other singularities of Ω, we construct the normalized solutions J_a, J_b analytically continued along γ and *the scattering operator*

$$\widetilde{J}_b^a = J_a^{-1} J_b \in \mathbf{C}\langle\langle A_V \rangle\rangle\,.$$

Its coefficients (as power series in (A_v)), by definition, are *regularized iterated integrals* of (ω_v). It turns out that $\widetilde{J}_b^a$ satisfy the general properties of the iterated integrals summarized in 3.2.

Example: Drinfeld's associator. Let $X = \mathbf{P}^1(\mathbf{C})$, $V = \{0, 1\}$,

$$\omega_0 = \frac{1}{2\pi i}\,\frac{dz}{z}, \quad \omega_1 = \frac{1}{2\pi i}\,\frac{dz}{z-1}\,.$$

Then

$$\Omega = A_0\omega_0 + A_1\omega_1$$

has poles at $0, 1, \infty$ with residues $A_0/2\pi i$, $A_1/2\pi i$, $-(A_0+A_1)/2\pi i$ respectively. Put $t_0 = z, t_1 = 1 - z$. Then $\widetilde{J}_0^1$ in our notation is the Drinfeld associator $\phi_{KZ}(A_0, A_1)$.

Example: modular generalization of multiple zeta values. Let Γ be a congruence subgroup of the modular group, $(f_v) :=$ a basis of Eisenstein series of weight 2 wrt Γ, $\{\omega_v =$ push forward of $f_v(z)dz\}$: 1–forms with logarithmic singularities at cusps on X_Γ. The space of such forms has the maximal possible dimension, because the difference of any two cusps has finite order in the Jacobian (cf. [**Elk90**]).

Regularized iterated integrals of Eisenstein series of weight two along geodesics between cusps provide a modular generalization of multiple zeta values.

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Max–Planck–Institut für Mathematik, Vivatsgasse 7 53111 Bonn, Germany and Department of Mathematics, Northwestern University 2033 Sheridan Road Evanston, IL 60208-2730, USA

E-mail address: manin@math.northwestern.edu