Arithmetic geometry

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Introduction

Classically, arithmetic is the study of rational or integral solutions of Diophantine equations. From a modern standpoint, this is a particular case of the study of schemes over algebraically nonclosed fields and more general commutative rings. The geometric viewpoint, dating back to ancient Greece, has been a source of inspiration to generations of mathematicians. The guiding principle is that

geometry determines arithmetic.

The tremendous power of this principle has been amply demonstrated in the works of Faltings on the Mordell conjecture and Wiles on Fermat's last theorem.

This volume grew out of the 2006 Clay Summer School held at the Mathematisches Institut of the University of Göttingen. The goal of the school was to introduce participants to the wealth of new techniques and results in arithmetic geometry. The first three weeks of the school were devoted to three main courses, covering curves, surfaces and higher-dimensional varieties, respectively; the last week was dedicated to more advanced topics. An important component of the school was a seminar focused on computational and algorithmic aspects of arithmetic geometry. The present proceedings volume reflects this structure.

Curves:

The main *geometric* invariant of a curve is its genus; the *arithmetic* is very different for curves of genus $0, 1$ and ≥ 2 respectively. In genus 0, we can answer, completely and effectively, whether or not a curve contains rational points and how these points are distributed. The theory of genus 1 curves is one of the richest subjects in mathematics, with spectacular recent theorems, e.g., modularity of elliptic curves over the rationals, and with many outstanding open questions, such as the Birch/Swinnerton-Dyer conjecture. In higher genus, the most fundamental result is the proof of the Mordell conjecture by Faltings, and the most challenging open question is to give an effective version of this result.

The lecture notes by Darmon cover the following topics:

- Faltings' proof of the Mordell Conjecture;
- Rational points on modular curves and Mazur's approach to bounding them;
- Rational points on Fermat curves and Wiles' proof of Fermat's Last Theorem;

- Elliptic curves and the Birch and Swinnerton-Dyer conjecture, following Gross-Zagier and Kolyvagin.

Contributions by Chapdelaine, Charollois, Dasgupta, Greenberg, Rebolledo, and Voight discuss more specialised topics that grew out of these lectures, such as

- Generalised Fermat equations (Chapdelaine);
- Merel's extension of Mazur's techniques to study rational points on modular curves over number fields, and the uniform boundedness conjecture for torsion of elliptic curves (Rebolledo);
- Natural generalisations of Fermat's Last Theorem due to Kraus and Halberstadt, building on Frey's approach (Charollois);
- CM points on modular curves and their applications to elliptic curves (Dasgupta, Voight);
- Shimura curves with a focus on computational aspects (Voight, Greenberg);
- Stark-Heegner points (Greenberg).

In addition, a paper by Manin treats modular symbols (which play an important role in Merel's proof of the uniform boundedness conjecture explained in Rebolledo's article) and discusses higher dimensional generalizations.

Surfaces:

The geometry of surfaces over the complex numbers is much more involved, and their birational classification was a milestone in algebraic geometry. Hassett's paper gives a thorough introduction to this classification over nonclosed fields, and its implications for Diophantine questions like the existence of rational points and weak approximation. It also touches on geometric descent constructions generalizing Fermat's descent (universal torsors) and algebraic approaches to these objects (Cox rings).

Harari's paper discusses non-abelian versions of descent, which have yielded new counterexamples to local-global principles for rational points on surfaces over number fields. Once rational points exist, one can ask whether they are Zariski dense and analyze their distribution with respect to heights; these questions are addressed, for both surfaces and higher-dimensional varieties, in Tschinkel's survey.

Vioreanu offers tantalizing computational evidence for conjectures about the algebraic structure of rational points on cubic surfaces. He explores whether all points can be generated from a small number using elementary geometric operations.

Higher-dimensional varieties:

Some of the most interesting higher-dimensional varieties from the arithmetic point of view are low-degree hypersurfaces and varieties closely related to algebraic groups: toric varieties, homogeneous spaces, and equivariant compactifications of groups. Here one is interested in existence questions, density of rational points, and counting points of bounded height. For the last problem, height zeta functions are an important tool and techniques of harmonic analysis can be profitably employed. A selection of recent results in this direction appears in the survey of Tschinkel.

Recent advances in higher-dimensional birational geometry over algebraically closed ground fields, within the framework of the Minimal Model Program, gave new impetus for arithmetic investigations. Abramovich introduces various classification theories, e.g., Kodaira's classification, Campana's classification, and related conjectures concerning Zariski density of rational points.

Historically, the Minimal Model Program led to the introduction of rationally connected varieties. The arithmetic of these varieties over function fields is the subject of Starr's paper. Some of the outstanding problems are: intrinsic geometric characterization of varieties that have points over every function field of a fixed transcendence degree, identification of cohomology obstructions to the existence of points on rationally connected varieties over the function field of a surface, and weak approximation over function fields of curves.

Abelian varieties and their moduli spaces have always played a central rôle in arithmetic geometry. Here one is often interested in the interaction between rational points and special subvarieties, e.g., translates of abelian subvarieties by torsion points, Shimura subvarieties etc. An important technique is to analyze equidistribution of points under Galois actions. This is illustrated in the paper of Ratazzi/Ullmo, which gives a new proof of the Manin-Mumford conjecture. This strategy is pushed further in the paper of Ullmo/Yafaev surveying new results on the Andre-Oort conjecture. Moduli spaces of abelian varieties over *finite fields* are represented by the lectures of Chai/Oort, addressing the density of Hecke orbits and p-divisible groups.

The survey of Kaledin, on the noncommutative analog of the Cartier isomorphism and Hodge Theory, illustrates the rich cross-fertilization between complex and characteristic-p algebraic geometry.

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