Math 212-4 - Practice Final Solutions

April 27, 2007

1. For material covered on the first two exams (chapters 1-6), review questions from the exams and practice exams. For material in chapters 7 and 8, here are some review questions:

2. Be able to state the Green's, Divergence, and Stoke's Theorem.

3. (a) Evaluate the integral $\int_C ydx - xdy$ where C is the boundary of the triangle with vertices $(0,0,0),(10,0,0),(3,5,0)$.

   We could do this directly, but since the boundary of the triangle consists of three edges, we would have to parameterize the three parts and evaluate the line integral along each one. Instead, we can use Green's Theorem to convert the line integral to an integral over the entire triangle. We have

   $$\int_C ydx - xdy = \int\int_D -1 - 1dA = -2\int\int_D dA$$

   But since $\int\int_D dA$ is just the area of the region D, which is the triangle, this will be $1/2 * 10 * 5 = 25$. Therefore

   $$\int_C ydx - xdy = -2\int\int_D dA = -2 * 25 = -50$$

   (b) Let R be the region in the plane given by $0 \leq x \leq 1, x^2 \leq y \leq x$. Evaluate the integral $\int_C x^2dx + xydy$, where C is the boundary of R.

   Again, we can use Green's Theorem to compute the line integral

   $$\int_C x^2dx + xydy = \int\int_D ydA = \int_0^1\int_{x^2}^x ydydx$$

   which when evaluated is

   $$\frac{1}{2}\int_0^1 x^2 - x^4dx = \frac{1}{2}(\frac{1}{3} - \frac{1}{5})$$

4. (a) Find the flux of the vector field $\mathbf{F}(x, y, z) = (x^2, z - 2xy, zx)$ out of the sphere of radius 5 centered at the origin, with the outward normal.

Finding the flux is calculating the surface integral

$$\int\int_S \mathbf{F} \cdot d\mathbf{S}$$

Because the surface in this problem is a closed sphere that completely encloses a solid 3D region, the solid ball, we can use the Divergence Theorem.

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} = \int\int\int_W div\mathbf{F}dV = \int\int\int_W xdxdydz$$

To evaluate this integral over the solid ball of radius 5, it is probably best to change to spherical coordinates. Then, we get

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{5}(\rho\sin\phi\cos\theta)\rho^2\sin\phi d\rho \; d\phi d\theta$$

which when evaluated should be 0.

5. Calculate the flux of $\mathbf{F}(x, y, z) = (x^3 + y\sin z, y^3 + z\sin x, z^3)$ across the closed surface bounded by the hemisphere $z = \sqrt{4 - x^2 - y^2}$, and the plane $z = 0$.

The surface in this problem is meant to include the cap to the hemisphere, but this isn't exactly clear in the wording. I will make sure to specify on the exam so there is no ambiguity. But, if the cap is included, again this surface completely bounds a 3D region, so we may use the Divergence Theorem:

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} = \int\int\int_W div\mathbf{F}dV = \int\int\int_W 3x^2 + 3y^2 + 3z^2dxdydz$$

Again, we'll need to change to spherical coordinates to get

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2} 3(\rho^2)\rho^2\sin\phi d\rho d\phi d\theta$$

which when evaluated should be $192\pi/5$.

Now, suppose instead the cap was not included in the surface S, and the question asks you to compute the surface integral of the curl $\mathbf{F}$ over S. In this case, the boundary of our surface S is a curve C defined by $x^2+y^2=4$. Now, in this case, we could use Stoke's Theorem:

$$\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{s}$$

which when we use $c(t) = (2\cos t, 2\sin t, 0)$ to parameterize C, we get

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} -16\cos^3 t \sin t + 16 \sin^3 t \cos t dt$$

$$= 4\cos^4 t + 4\sin^4 t|_0^\pi = 0$$

6. (a) Find the area of the surface defined by $\mathbf{\Phi}(u,v) = (u+v, u, v), u \in [0,1], v \in [0,1]$.

   Surface Area =

$$\int\int_S dS = \int_0^1 \int_0^1 ||T_u \times T_v|| du dv$$

   $T_u = (1,1,0), T_v = (1,0,1), T_u \times T_v = (1,-1,-1)$. Then $||T_u \times T_v|| = \sqrt{3}$. Thus, the surface area is

$$\int_0^1 \int_0^1 \sqrt{3} du dv = \sqrt{3}$$

   (b) Compute the integral of $f = x^2+y^2+z^2$ over the surface in part a).

$$\int\int_S f dS = \int_0^1 \int_0^1 f(\Phi(u,v))||T_u \times T_v|| du dv = \sqrt{3} \int_0^1 \int_0^1 2u^2 + 2v^2 + 2uv du dv$$

   (c) Calculate the unit normal to the surface.

   Based on what we've already calculated, the unit normal is

$$\frac{T_u \times T_v}{||T_u \times T_v||} = (1,-1,-1)/\sqrt{3}$$

7. Evaluate $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ where $\mathbf{c}(t) = (\cos^5 t, \sin^3 t, t^4), t \in [0,\pi]$, and $\mathbf{F}(x,y,z) = (2xyz + \sin x, x^2 z, x^2 y)$. Hint: Is $\mathbf{F}$ a conservative vector field?

   To test whether or not $\mathbf{F}$ is a conservative vector field, it is enough to test whether the curl is zero. It is a simple calculation to show that the curl is actually zero for this $\mathbf{F}$. Therefore, there exists an f so that $\nabla f = \mathbf{F}$. We can solve for this f by solving $f_x = 2xyz + \sin x, f_y = x^2 z, f_z = x^2 y$ and the integrating each part to find that $f = x^2 yz - \cos x$.

Thus, we can evaluate the line integral by evaluating f at the endpoints.

$$\int_c \mathbf{F} \cdot d\mathbf{s} = f(c(\pi)) - f(c(0))$$

$$= f(-1, 0, \pi^4) - f(1, 0, 0) = -\cos(-1) - \cos(1) = 0$$