Math 212-4 - Practice/Review Questions for Exam 2

March 15, 2007

1. Make indicated change of variables, but do not evaluate.

   (a) $\int_0^1 \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \sqrt{x^2+y^2}dxdydz$, cylindrical coordinates

   $$\int_0^1 \int_0^1 \int_0^{2\pi} r^2 d\theta drdz$$

   (b) $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} xyzdzdxdy$, cylindrical coordinates

   $$\int_0^{2\pi} \int_0^1 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r^3 \cos\theta \sin\theta zdzdrd\theta$$

   (c) $\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} z^2 dzdxdy$, spherical coordinates
   The region is bounded by the sphere $x^2+y^2+z^2=4$ and the cone $z^2=x^2+y^2$. The mistake I made, was in calculating the bounds for ρ. Recall that ρ is the length of the distance from any point in the region to the origin. The largest ρ can be is when we are on the sphere, so $\rho=2$. However, the smallest ρ can be is zero, since the origin is contained in our region. Now the bounds for ϕ determine the boundary of the cone. Since ϕ is the angle with the z-axis, ϕ can go from zero to the edge of the cone, which occurs at $\phi=\pi/4$. ($x^2+y^2=z^2$ is $\rho^2\sin^2\phi=\rho^2\cos^2\phi$, which will only happen when $\phi=\pi/4$). Thus, our integral in spherical coordinates is

   $$\int_0^{2\pi} \int_0^{\pi/4} \int_0^2 \rho^4 \cos^2\phi \sin\phi d\rho d\phi d\theta$$

   (d) $\int_0^1 \int_0^{\pi/4} \int_0^{2\pi} \rho^3 \sin 2\phi d\theta d\phi d\rho$, rectangular
   This is almost like the previous problem. Recall that $\sin(2\phi) = 2\sin\phi\cos\phi$, so the function $\rho^3\sin(2\phi)d\theta d\phi d\rho = 2zdzdydx$. Now, we have the region bounded by the sphere of radius one and the cone

$x^2 + y^2 = z^2$. The intersection of these two surfaces happens when $z = 1/\sqrt{2}$ in a circle of radius $1/\sqrt{2}$. Thus, our bounds are $\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1/2-y^2}}^{\sqrt{1/2-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} z^2 dz dx dy$,

2. Evaluate
$$\int_0^1 \int_{\arctan(y)}^{\pi/4} \sec^5(x) dx dy$$
Change the order of integration to get
$$\int_0^{\pi/4} \int_0^{\tan x} \sec^5(x) dy dx$$
and evaluate to get the answer $\frac{4\sqrt{2}-1}{5}$

3. Write the iterated integral
$$\int_0^1 \int_{1-x}^1 \int_x^1 f(x,y,z) dz dy dx$$
as an equivalent integral written in two other possible orders of integration.

An easy switch is to change the order of x and y. After sketching the region in the xy plane, the proper endpoints are
$$\int_0^1 \int_{1-y}^1 \int_x^1 f(x,y,z) dz dx dy$$
To switch z with x or y is a bit more difficult. You can sketch the region (which is harder than I first thought) and see that two other possible orders of integration are
$$\int_0^1 \int_x^1 \int_{1-x}^1 f(x,y,z) dy dz dx$$
and
$$\int_0^1 \int_{1-z}^1 \int_{1-y}^z f(x,y,z) dx dy dz$$
An easier way is to notice that bounds of z do not depend on y. Thus, these two may be switched without change (first of two above).

4. Find the volume of the solid between the paraboloid $z = 3x^2 + 3y^2$ and $z = 12 - 3x^2 - 3y^2$.

   The two paraboloids intersect in a circle $x^2 + y^2 = 2$ when $z = 6$. In rectangular coordinates, an integral for the volume is

$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{3x^2+3y^2}^{12-3x^2-3y^2} dzdydx$$

   In polar coordinates this becomes

$$\int_0^{2\pi} \int_0^{\sqrt{2}} \int_{3r^2}^{12-3r^2} rdzdrd\theta$$

   Evaluating this integral, $V = 12\pi$.

5. Find the volume of the solid region (ice cream cone) that lies inside the sphere $x^2 + y^2 + z^2 = z$ and above the cone $z^2 = x^2 + y^2, z \geq 0$.

   Rewrite the equation for the sphere as $x^2 + y^2 + (z - 1/2)^2 = 1/4$, so this is a sphere of radius $1/2$ centered at $(0, 0, 1/2)$.

   The sphere and cone intersect when $z = 1/2$ in a circle of radius $1/2$.

   To set up the volume in a single integral, we could use cylindrical coordinates. The sphere $x^2+y^2+(z-1/2)^2 = 1/4$ becomes $z = \sqrt{1/4 - r^2}+1/2$. The cone $x^2 + y^2 = z^2$ becomes $z = r$. Thus

$$V = \int_0^{2\pi} \int_0^{1/2} \int_r^{\sqrt{1/4-r^2}+1/2} rdzdrd\theta = \pi/8$$

   We could also use spherical coordinates. For the sphere $z^2 + y^2 + x^2 = z$ becomes $\rho^2 = \rho \cos \phi$, which gives us $\rho = \cos \phi$. Now, the cone gives us the bounds for ϕ. Using $z = \sqrt{x^2 + y^2}$, $\rho \cos \phi = \rho \sin \phi$, which must mean $\phi = \pi/4$. Therefore

$$\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos \phi} \rho^2 \sin \phi d\rho d\phi d\theta$$

   which when evaluated also yields the answer $\pi/8$.

6. Show that $\mathbf{c}(t) = (t^{-3}, e^t, t^{-1})$ is a flow line of the vector field $\mathbf{F}(x, y, z) = (-3z^4, y, -z^2)$.

   $c'(t) = (-3t^{-4}, e^t, -t^{-2})$. $F(c(t)) = (-3t^{-4}, e^t, -t^{-2})$. Since $c'(t) = F(c(t))$, c is a flow line.

7. Compute divergence and curl for the vector field $\mathbf{F}(x, y, z) = (y, z, x)$ at the point $(1, 1, 1)$.

   div $F = 0 + 0 + 0 = 0$.

   curl $F = (-1, -1, -1)$ for and point. Thus at $(1, 1, 1)$, div $F = 0$, curl $F = (-1, -1, -1,)$.

8. Calculate the arc length of the curve defined by $x = y^3 = z^2 + 1$ from the $x = 0$ to $x = 1$.

   If let $y = t$, $x = t^3$, $z = \sqrt{t^3 - 1}$. $c(t) = (t^3, t, \sqrt{t^3 - 1}$, $c'(t) = (3t^2, 1, 3/2t^2(t^3 - 1)^{-1/2})$, $||c'(t)|| = (9t^4 + 1 + 9/4t^4(t^3 - 1)^{-1}$. Then the arc length will be

   $$\int_0^1 (9t^4 + 1 + 9/4t^4(t^3 - 1)^{-1} dt$$

   which would need to be looked up in a table to calculate explicitly.