

Math 212-4 - Final

April 29, 2007

Instructions: You have **3 hours** to complete the exam. Write the time you start and end on the exam, along with the honor code. You may NOT use any books, notes, calculator, or other people. Do not discuss the exam with anyone except your instructor until all exams are turned in.

There are 10 questions. Show all work to receive full credit. Clearly cite any theorems you use (for example, Green's Theorem, etc.). Partial credit will be given, so it is best to turn in all work. Indicate final answer clearly. The exam will be due **May 7th, 5 pm**. Good luck!

Formulas you can use:

1. Cylindrical coordinates: $x = r \cos \theta, y = r \sin \theta, z = z, 0 \leq r, 0 \leq \theta \leq 2\pi$.
2. Spherical coordinates: $x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, z = \rho \cos \phi, 0 \leq \rho, 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi$.

1. Answer T/F. No explanation is needed.
 - (a) For vectors \mathbf{u}, \mathbf{v} , if $\mathbf{u} \cdot \mathbf{v} = 0$, then \mathbf{u}, \mathbf{v} are parallel.
 - (b) The point $(0, 1/\sqrt{2}, 1/\sqrt{2})$ in rectangular coordinates is $(1, \pi/4, \pi/2)$ in spherical coordinates.
 - (c) Level sets for the function $f(x, y) = \sqrt{x^2 + y^2}$ are concentric circles.
 - (d) Path integrals remain unchanged when the orientation of the curve is reversed.
 - (e) Line integrals remain unchanged when the orientation of the curve is reversed.
 - (f) $\mathbf{F}(x, y, z) = ((2x + x^2y)e^{xy}, x^3e^{xy}, 1)$ is a conservative vector field.
 - (g) For \mathbf{F} above (in (f)), $\int_C \mathbf{F} \cdot d\mathbf{s} = 0$ for any closed curve C .
 - (h) For any vector field $\mathbf{F}(x, y, z)$, $\text{div } \mathbf{F} = \nabla \cdot \mathbf{F}$ is a vector.
 - (i) For any vector field $\mathbf{F}(x, y, z)$, $\text{curl } \mathbf{F} = \nabla \times \mathbf{F}$ is a vector.
 - (j) $\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = 0$ if S is the unit sphere for any C^1 vector field \mathbf{F} (Hint: Use a theorem).

2. Evaluate the integral

$$\int_0^1 \int_{\sqrt{y}}^1 e^{x^3} dx dy$$

3. $g(x, y) = x^2y + 2xy$. \mathbf{h} is a differentiable function, $\mathbf{h} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $\mathbf{h}(1, 1) = (2, 1)$,

$$D\mathbf{h}(1, 1) = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}.$$

Let $f(x, y) = g \circ \mathbf{h} = g(\mathbf{h}(x, y))$. Find $Df(1, 1)$.

4. Let $f(x, y) = e^{xy} \sin(x + y)$.
 - (a) In what direction, starting at $(0, \pi/2)$ is f changing the fastest?
 - (b) Let $\mathbf{c}(t)$ be a flow line of $\mathbf{F} = \nabla f$ with $\mathbf{c}(0) = (0, \pi/2)$. Calculate $\frac{d}{dt}[f(\mathbf{c}(t))]$ evaluated at $t = 0$.
5. $f(x, y) = x^2 + xy + y^2$
 - (a) Find the maximum of f on the circle $x^2 + y^2 = 1$.
 - (b) Find the maximum value of f in the disk $x^2 + y^2 \leq 1$.
6. Let W be the region in space under the graph of

$$f(x, y) = \cos(y)e^{1-\cos(2x)} + xy$$

and over the region in the xy -plane bounded by the line $y = 2x, x = 0, x = \pi/4$.

- (a) Find the volume of W .
- (b) Let $\mathbf{F}(x, y, z) = (5x, 5y, 5z)$ be the velocity field of a fluid in space. Calculate the flux of \mathbf{F} through the boundary, ∂W , of W ($\int \int_{\partial W} \mathbf{F} \cdot d\mathbf{S}$), where W is the region from a). (Hint: What is the divergence of \mathbf{F} ?)
7. $\mathbf{F}(x, y, z) = (2yz, -x + 3y + 2, x^2 + z)$.
- (a) Evaluate $\int \int_S (\nabla \times F) \cdot d\mathbf{S}$ where S is the cylinder $x^2 + y^2 = a^2$, $0 \leq z \leq 1$, but not including the top or bottom caps of the cylinder. (You may find the following identity helpful: $\sin^2 t = \frac{1 - \cos(2t)}{2}$).
- (b) Redo part a) if the top and bottom of cylinder are included in the surface S . (You shouldn't have to calculate much for this part!)
8. Evaluate $\int_C x^3 dy - y^3 dx$ where C is the boundary of the unit square $[0, 1] \times [0, 1]$.
9. A paraboloid of revolution S is parameterized by $\Phi(u, v) = (u \cos v, u \sin v, u^2)$, $u \in [0, 2]$, $v \in [0, 2\pi]$.
- (a) Find a unit normal vector to the surface $\Phi(u, v)$.
- (b) Find equation for the tangent plane at $\Phi(u_0, v_0) = (1, 1, 2)$.
- (c) Find the surface area of S .
10. Evaluate $\int_C \sin(\pi x) dy - \cos(\pi y) dz$ where C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$.