Total variation regularization for image denoising: Theory and Examples.

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Let Ω be an open subset of $\mathbf{R}^n$ where $2 \le n \le 7$; the reason for restriction on n is that our work will make use of the regularity theory for area minimizing hypersurfaces.

Let $s \in \mathbf{L}_1(\Omega) \cap \mathbf{L}_\infty(\Omega)$, let $\gamma : \mathbf{R} \to [0, \infty)$ be zero at zero, nondecreasing and smooth on $[0, \infty)$ and convex and, for $f \in \mathbf{L}_1(\Omega)$, let

$$F(f) = \int_\Omega \gamma(f(x) - s(x))\, d\mathcal{L}^n x;$$

$\mathcal{L}^n$ here is Lebesgue measure on $\mathbf{R}^n$. In the denoising literature F would be called a **fidelity** term in that it measures deviation from s which could be a noisy grayscale image. Let $\epsilon > 0$ and, for $f \in \mathbf{L}_\infty(\Omega)$, let

$$F_\epsilon(f) = \epsilon \mathbf{TV}(f) + F(f);$$

here $\mathbf{TV}(f)$ is the total variation of f. A minimizer of F_ϵ is called a **total variation regularization (TVR) of** s. Rudin, Osher and Fatemi and Chan and Esedoglu have studied TVRs of F where $\gamma(y) = y2$ and $\gamma(y) = y$, $y \in \mathbf{R}$, respectively. In applications n is typically $2, 3$ or 4 and f as above is thought to be a "denoising" of s.

Let f be a TVR of s. The first main result of this work is that the reduced boundaries of the sets $\{f \ge y\}$, $y \in \mathbf{R}$ are $C^{1+\mu}$ hypersurfaces for any $\mu \in (0, 1)$ in case $n > 2$ and any $\mu \in (0, 1]$ in case $n = 2$; moreover, the generalized mean curvature of the sets $\{f \ge y\}$ will be bounded by constants one can readily determine from the essential supremum of $|s|$. In fact, this result holds for a rather general class of fidelities. A second result gives precise curvature information about the reduced boundary of $\{f \ge y\}$ near points where s is smooth. This curvature information will allow us to construct a number of interesting examples of TVRs. In addition to providing insight as to the nature of TVRs, these examples may be used to validate computational schemes which purport to approximate TVRs.