Homework 8, due Friday 3/10

1. Let X be a topological space with path connected components $X_1, \dots, X_n$ and let G be an abelian group. Show that there is an isomorphism

$$\tilde{H}^0(X;G) \cong \{\varphi : X \to G \text{ constant on each } X_i\}/\{\varphi : X \to G \text{ constant on } X\}.$$

2. Let $\varphi : A \to B$ be a homomorphism of abelian groups. Show that φ induces a functor (covariant or contravariant?) on homology groups with coefficients and cohomology with coefficients. (A sketch of the argument is enough here).

3. Let

$$0 \to A \to B \to C \to 0$$

   be an exact sequence of abelian groups.

   (a) Let G be another abelian group. Show that

$$\mathrm{Hom}(C,G) \to \mathrm{Hom}(B,G) \to \mathrm{Hom}(A,G) \to 0$$

   is exact (which is why $\mathrm{Hom}(-,G)$ is called a right exact functor).

   (b) Give an example of a short exact sequence with $G = \mathbb{Z}/2$ where

$$0 \to \mathrm{Hom}(C,G) \to \mathrm{Hom}(B,G) \to \mathrm{Hom}(A,G) \to 0$$

   is not exact.

4. Let $C_2 = \mathbb{Z}, C_1 = \mathbb{Z}^3, C_0 = \mathbb{Z}, C_i = 0$ for $i \in \mathbb{Z} \setminus \{0,1,2\}$. Let $\partial_2 : C_2 \to C_1$ be the map given by the matrix $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ and let $\partial_1 : C_1 \to C_0$ be the map given by the matrix $\begin{pmatrix} 2 & 0 & 4 \end{pmatrix}$. Furthermore let $\partial_i : C_i \to C_{i-1}$ be the zero map for $i \neq 1,2$. In Homework 1 you already showed that $\partial_i \circ \partial_{i+1} = 0$ for all i.

   (a) Determine the homology of the complex $\mathrm{Hom}(C_*;\mathbb{Z})$.

   (b) Determine the homology of the complex $\mathrm{Hom}(C_*;\mathbb{Z}/2)$.

   (c) Determine the homology of the complex $C_* \otimes \mathbb{Z}/2$.