Homework 8, due Friday 3/10

1. Let $X$ be a topological space with path connected components $X_1, \ldots, X_n$ and let $G$ be an abelian group. Show that there is an isomorphism
\[
\tilde{H}^0(X; G) \cong \{ \varphi : X \to G \text{ constant on each } X_i \}/\{ \varphi : X \to G \text{ constant on } X \}.
\]

2. Let $\varphi : A \to B$ be a homomorphism of abelian groups. Show that $\varphi$ induces a functor (covariant or contravariant?) on homology groups with coefficients and cohomology with coefficients. (A sketch of the argument is enough here).

3. Let
\[
0 \to A \to B \to C \to 0
\]
be an exact sequence of abelian groups.

(a) Let $G$ be another abelian group. Show that
\[
\text{Hom}(C, G) \to \text{Hom}(B, G) \to \text{Hom}(A, G) \to 0
\]
is exact (which is why $\text{Hom}(-, G)$ is called a right exact functor).

(b) Give an example of a short exact sequence with $G = \mathbb{Z}/2$ where
\[
0 \to \text{Hom}(C, G) \to \text{Hom}(B, G) \to \text{Hom}(A, G) \to 0
\]
is not exact.

4. Let $C_2 = \mathbb{Z}, C_1 = \mathbb{Z}^3, C_0 = \mathbb{Z}, C_i = 0$ for $i \in \mathbb{Z} \setminus \{0, 1, 2\}$. Let $\partial_2 : C_2 \to C_1$ be the map given by the matrix $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ and let $\partial_1 : C_1 \to C_0$ be the map given by the matrix $\begin{pmatrix} 2 & 0 & 4 \end{pmatrix}$.

Furthermore let $\partial_i : C_i \to C_{i-1}$ be the zero map for $i \neq 1, 2$. In Homework 1 you already showed that $\partial_i \circ \partial_{i+1} = 0$ for all $i$.

(a) Determine the homology of the complex $\text{Hom}(C_\ast; \mathbb{Z})$.

(b) Determine the homology of the complex $\text{Hom}(C_\ast; \mathbb{Z}/2)$.

(c) Determine the homology of the complex $C_\ast \otimes \mathbb{Z}/2$. 