MATH 445: Algebraic Topology **Spring 2006**

Homework 7, due Friday 3/3

1. Use the definition of the tensor product of abelian groups to show that $\mathbb{Z}/n\mathbb{Z} \otimes A \cong A/nA$, $\mathbb{Z}/n \otimes \mathbb{Q} = 0$ and $\mathbb{Z} \otimes A \cong A$ for any abelian group A.

2. Let $\mathcal{A}$ be the category of finitely generated abelian groups.

   (a) Let $\mathcal{F}$ be the category of finitely generated free abelian groups. Show that there are functors $F : \mathcal{A} \to \mathcal{F}$ and $G : \mathcal{F} \to \mathcal{A}$ such that $F \circ G$ is the identity functor on $\mathcal{F}$.

   (b) Now let $\mathcal{G}$ be the category of finite abelian groups. Are there functors $F : \mathcal{A} \to \mathcal{G}$ and $G : \mathcal{G} \to \mathcal{A}$ such that $F \circ G$ is the identity functor on $\mathcal{G}$?

3. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of abelian groups. Let G be an abelian group as well. Show that $A \otimes G \to B \otimes G \to C \otimes G \to 0$ is a short exact sequence.

4. Give an example of a short exact sequence of abelian groups $0 \to A \to B \to C \to 0$ and of an abelian group G such that $0 \to \hom(C, G) \to \hom(B, G) \to \hom(A, G) \to 0$ is not a short exact sequence.

5. Let $\mathcal{C}$ be the category of pairs (X, A) where X is a topological space and $A \subset X$. Let $\mathcal{A}$ be the category of finitely generated abelian groups. Consider the two functors $F : X \mapsto H_n(X)$ and $G : X \mapsto H_n(X, A)$. Give an example of a non–trivial natural transformation between F and G.

6. Show that for any finitely generated abelian group A we have $\mathrm{Tor}(A, \mathbb{Q}) = 0$.