Homework 3, due Friday 2/3

1. Assume $A$ is an abelian group which fits into an exact sequence

\[ 0 \to \mathbb{Z}/p \to A \to \mathbb{Z}/q \to 0, \]

where $p, q$ are prime numbers.

(a) For which $p, q$ is $A$ uniquely determined?

(b) Determine the possible answers for $A$.

2. Let $\mathcal{F}$ be a field. Assume we have an exact sequence

\[ 0 \to \mathcal{F}^n \to \mathcal{F}^{n-1} \to \cdots \to \mathcal{F}^1 \to \mathcal{F}^0 \to 0 \]

of homomorphisms.

(a) Show that

\[ 0 \to \mathcal{F}^n \to \mathcal{F}^{n-1} \to \mathcal{F}^{n-2} \cdots \to \text{Im}(\mathcal{F}^2 \to \mathcal{F}^1) \to 0 \]

is exact.

(b) Show that $\sum_{i=0}^{k} (-1)^i n_i = 0$.

3. Let $0 \to A_\ast \to B_\ast \to C_\ast \to 0$ be a short exact sequence of complexes. Show that if $A_\ast$ and $B_\ast$ are exact, then $C_\ast$ is exact as well.


5. Let $0 \to A_\ast \xrightarrow{i} B_\ast \xrightarrow{\pi} C_\ast \to 0$ be a short exact sequence of complexes. Proof the following inclusions without looking at the book (except for reading the definition of the connecting homomorphism):

(a) $\text{Ker}\{\partial : C_n \to C_{n-1}\} \subset \text{Im}\{\pi : B_n \to C_n\}$.

(b) $\text{Ker}\{i : A_n \to B_n\} \subset \text{Im}\{\partial : C_{n+1} \to A_n\}$.

6. Let $I = [0, 2]$. Show that $H_n(I, \{0\} \cup \{1\} \cup \{2\}) = 0$ for $n > 1$ and $n = 0$ and that $H_n(I, \{0\} \cup \{1\} \cup \{2\}) \cong \mathbb{Z}^2$. 