Homework 13, due Friday 4/28

1. Read the definition of the topological structure of $\tilde{M}$ on p. 234.

2. (a) Show that the Euler characteristic of an odd–dimensional closed manifold is zero.
   (b) Let M be a compact 3–manifold such that its boundary components are all tori. Show that the Euler characteristic of M is zero.

3. Let M be a connected closed n–manifold. Show that the following two statements are equivalent:
   (a) For every $x \in M$ the map $H_n(M;R) \to H_n(M, M \setminus x; R)$ is an isomorphism.
   (b) There exists an $x \in M$ such that the map $H_n(M;R) \to H_n(M, M \setminus x; R)$ is an isomorphism.

4. Let M be a closed 3–manifold with $H_1(M;\mathbb{Z})$ finite. Show that M is a rational homology sphere, i.e. $H_*(M;\mathbb{Q}) = H_*(S^3;\mathbb{Q})$.

5. Let M be a closed 3–manifold with $H_1(M;\mathbb{Z})$ finite. Assume that M is the boundary of a compact 4–manifold W with $H_i(W;\mathbb{Q}) = 0$ for $i \geq 1$.
   (a) Show that there exists an exact sequence
   $$0 \to A \to B \to H_1(M;\mathbb{Z}) \to B \to A \to 0,$$
   with A and B finite groups. (Hint: Consider the long exact sequence for the pair (W, M), then use Poincaré duality and the universal coefficient theorem).
   (b) Show that $|H_1(M;\mathbb{Z})|$ is a square.

6. Let M be a $2n$–dimensional closed manifold. Assume that M has a CW–structure such that there exists a cell in every dimension less than n. Does it follow that M has a cell in every dimension greater than n?