MATH 445: Algebraic Topology **Spring 2006**

Homework 10, due Friday 3/31

1. Show that there exists a natural transformation between the functors $X \mapsto H^k(X;\mathbb{Q})$ and $X \mapsto \mathrm{Hom}(H_K(X;\mathbb{Q}),\mathbb{Q})$ and that it has an inverse. (first think about why the groups are always isomorphic, then read the theorems more carefully for the 'natural' part of the statement).

2. Let X be a path connected space and $A \subset X$ a subset. Show that $H^0(X, A) = 0$ without using the Universal Coefficient Theorem.

3. Define a cup product $C^k(X;R) \times C^l(X,A;R) \to C^{k+l}(X,A;R)$ and show that it induces a well–defined map $H^k(X;R) \times H^l(X,A;R) \to H^{k+l}(X,A;R)$.

4. Let M be a connected, compact, oriented m–manifold without boundary. Let $D \subset M$ be a closed ball, show that $H_i(M) = H_i(M \setminus \mathrm{int}(D))$ for $i < m$ and $H_m(M \setminus \mathrm{int}(D)) = 0$. You can use the fact from class that $H_i(M) = \mathbb{Z}$.

5. Consider the torus T with the following two curves: Show that a and b generate $H_1(T)$.

   (use the previous problem, think about what you get once you remove a disk away from a and b).

6. Now let S be the oriented closed surface of genus g. Give S the orientation 'pointing outward'. Think about why the following curves $a_1, \ldots, a_g, b_1, \ldots, b_g$ form a basis for $H_1(S)$: Now, very

   informally, determine the map $H^1(S;\mathbb{Z}) \times H^1(S;\mathbb{Z}) \to H^2(S;\mathbb{Z}) \cong \mathbb{Z}$.