Homework 7, due Friday 3/3

2. Let R be the region given by removing the rectangle with vertices $(1,0),(1,2),(-1,2),(-1,0)$ from the circle with radius 4 around the origin. Compute $\int\int_R xy\,dA$.
Solution: It's easiest to first compute the integral of xy over the disk with radius 4, then compute the integral of xy over the rectangle, and then subtract.
Let's first do the disk, we compute

$$\int_{x=-4}^{x=4}\int_{y=-\sqrt{16-x^2}}^{y=\sqrt{16-x^2}} xydydx = \int_{x=-4}^{x=4}[x\frac{y^2}{2}]_{y=-\sqrt{16-x^2}}^{y=\sqrt{16-x^2}}dx.$$

(of course you could also do polar coordinates) But we see that $[x\frac{y^2}{2}]_{y=-\sqrt{16-x^2}}^{y=\sqrt{16-x^2}}$ is in fact zero, so the whole double integral is zero.
Now let's do the rectangle, we compute:

$$\begin{aligned}\int_{x=-1}^{x=1}\int_{y=0}^{y=2} xy\,dydx &= \int_{x=-1}^{x=1}[x\frac{y^2}{2}]_{y=0}^{y=2}\,dx\\ &= \int_{x=-1}^{x=1} 2x\,dx\\ &= [x^2]_{x=-1}^{x=1}\\ &= 0.\end{aligned}$$

So the integral over R is the first integral minus the second integral, so it equals zero.

4. p. 364, problem 9 (recall that the volume equals $\int\int\int 1\,dV$). Solution: I fear that my hint was perhaps not leading into the correct direction. The correct way of thinking about it is to look where the two graphs given by $z=x^2+y^2$ and $z=10-x^2-2y^2$ intersect. So we set them equal and we get

$$x^2+y^2=10-x^2-2y^2 \Rightarrow 2x^2+3y^2=10.$$

So the intersect in an ellipse. From this we see that the possible x–values are (by setting $y=0$) $x\in[-\sqrt{5},\sqrt{5}]$, furthermore given x the possible y–values are $y\in[-\sqrt{\frac{1}{3}(10-2x^2)},\sqrt{\frac{1}{3}(10-2x^2)}]$. Furthermore given x and y the possible z–values are given by $z\in[x^2+y^2, z=10-x^2-2y^2]$. Summarizing we get that the volume of the 3–dimensional body is given by

$$\int_{x=-\sqrt{5}}^{\sqrt{5}}\int_{y=-\sqrt{\frac{1}{3}(10-2x^2)}}^{y=\sqrt{\frac{1}{3}(10-2x^2)}}\int_{z=x^2+y^2}^{z=10-x^2-2y^2} 1\,dzdydx.$$

Note that many other descriptions are possible. We now compute

$$\int_{x=-\sqrt{5}}^{\sqrt{5}} \int_{y=-\sqrt{\frac{1}{3}(10-2x^2)}}^{y=\sqrt{\frac{1}{3}(10-2x^2)}} \int_{z=x^2+y^2}^{z=10-x^2-2y^2} 1\, dzdydx$$

$$= \int_{x=-\sqrt{5}}^{\sqrt{5}} \int_{y=-\sqrt{\frac{1}{3}(10-2x^2)}}^{y=\sqrt{\frac{1}{3}(10-2x^2)}} [z]_{z=x^2+y^2}^{z=10-x^2-2y^2} 1\, dzdydx$$

$$= \int_{x=-\sqrt{5}}^{\sqrt{5}} \int_{y=-\sqrt{\frac{1}{3}(10-2x^2)}}^{y=\sqrt{\frac{1}{3}(10-2x^2)}} 10 - 2x^2 - 3y^2\, dydx$$

$$= \int_{x=-\sqrt{5}}^{\sqrt{5}} [(10-2x^2)y - y^3]_{y=-\sqrt{\frac{1}{3}(10-2x^2)}}^{y=\sqrt{\frac{1}{3}(10-2x^2)}} dx$$

$$= \int_{x=-\sqrt{5}}^{\sqrt{5}} (10-2x^2)\sqrt{\tfrac{1}{3}(10-2x^2)} - (\sqrt{\tfrac{1}{3}(10-2x^2)})^3\, dx$$

Now one looks up some formulas to continue with the computation of the integral.

5. p. 364, problem 10 (use first equation to get the x–limits and the y–limits in terms of x, then use the remaining two equations to get the z–limits in terms of x and y).
Solution: From $x^2 + 2y^2 = 2$ we see that the possible x–values are $x \in [-\sqrt{2}, \sqrt{2}]$ and given x, the possible y–values are $[-\sqrt{1-\frac{x^2}{2}}, \sqrt{1-\frac{x^2}{2}}]$. Furthermore given x, y and the possible z–values are $z \in [0, 1 - \frac{1}{2}(x+y)]$. So the volume is

$$\int_{x=-\sqrt{2}}^{x=\sqrt{2}} \int_{y=-\sqrt{1-\frac{x^2}{2}}}^{y=\sqrt{1-\frac{x^2}{2}}} \int_{z=0}^{z=1-\frac{1}{2}(x+y)} 1\, dzdydx$$

$$= \int_{x=-\sqrt{2}}^{x=\sqrt{2}} \int_{y=-\sqrt{1-\frac{x^2}{2}}}^{y=\sqrt{1-\frac{x^2}{2}}} [z]_{z=0}^{z=1-\frac{1}{2}(x+y)}\, dydx$$

$$= \int_{x=-\sqrt{2}}^{x=\sqrt{2}} \int_{y=-\sqrt{1-\frac{x^2}{2}}}^{y=\sqrt{1-\frac{x^2}{2}}} 1 - \tfrac{1}{2}(x+y)\, dydx$$

$$= \int_{x=-\sqrt{2}}^{x=\sqrt{2}} [y - \tfrac{xy}{2} - \tfrac{y^2}{4}]_{y=-\sqrt{1-\frac{x^2}{2}}}^{y=\sqrt{1-\frac{x^2}{2}}}\, dx$$

$$= \int_{x=-\sqrt{2}}^{x=\sqrt{2}} 2\sqrt{1-\tfrac{x^2}{2}} - x\sqrt{1-\tfrac{x^2}{2}}\, dx.$$

Now break up the integral. The first part is $\int_{x=-\sqrt{2}}^{x=\sqrt{2}} 2\sqrt{1-\frac{x^2}{2}} = \int_{x=-\sqrt{2}}^{x=\sqrt{2}} \sqrt{2}\sqrt{2-x^2}$ is just $\sqrt{2}$ times the area of half the disk of radius $\sqrt{2}$. The second integral ($\int x\sqrt{1-\frac{x^2}{2}}\, dx$) can be done by substitution.

6. p. 364, problem 11.
The problem is: Find the volume of the region bounded by $x = y, z = 0, y = 0, x = 1$ and $x + y + z = 0$.
Solution: The hint should have said: assume that $x, y \geq 0$. The region is a pyramid between the positive x–axis and the positive y–axis, but below the xy–plane. The limits are

$$\begin{array}{rcl} 0 & \leq x \leq & 1 \\ 0 & \leq y \leq & x \text{ from } y = 0 \text{ and } x = y \\ -x - y & \leq z \leq & 0 \text{ from } z = 0 \text{ and } x + y + z = 0, \text{ note that } -x - y \leq 0. \end{array}$$

So the volume of the region is given by

$$\int_{x=0}^{x=1} \int_{y=0}^{y=x} \int_{z=-x-y}^{z=0} 1 \, dzdydx.$$

8. p. 364, problem 16.
Solution: First draw a careful picture, what you see is the quarter of a cylinder where the base has radius 1. In particular $x \in [0, 1]$, $y \in [0, \sqrt{1 - x^2}]$, $z \in [0, 1]$. And we get

$$\begin{array}{rcl} \int\int\int_W z \, dV & = & \int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{1-x^2}} \int_{z=0}^{z=1} z \, dzdydx \\ & = & \int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{1-x^2}} [\frac{z^2}{2}]_{z=0}^{z=1} \, dydx \\ & = & \int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{1-x^2}} \frac{1}{2} \, dydx. \end{array}$$

But now we are integrating a constant over the quarter of a disk of radius one. So the integral equals $\frac{1}{2}\pi$.