Homework 6 – solutions

2. Find the absolute maximum and minimum of $f(x, y) = 3x+2y$ in the disk $\{(x, y)|x^2+y^2 \leq 4\}$ of radius 2.

   Solution: This problem breaks into two pieces. We first have to find all critical points lying in the disk, and then we have to find the maximum and minimum on the boundary.

   (a) We compute $f_x = 3, f_y = 2$, these are never zero, so $f(x, y)$ has no critical point.

   (b) Now we have to find the absolute maximum and minimum of $f(x, y) = 3x + 2y$ on the *circle* $\{(x, y)|x^2 + y^2 = 2\}$. There are two possible ways to approach this problem. We can either parametrize the circle (i.e. describe as a path $\mathbf{c}(t)$) or we can use Lagrange multipliers. It doesn't really matter which approach we take in this case. Let's use Lagrange multipliers for practice.

   We describe the circle as a level set of the function $g(x, y) = x^2 + y^2$. More precisely, the circle is all points (x, y) such that $g(x, y) = 2$. We now have to find x, y, λ such that

   $$\nabla f(x, y) = \lambda \nabla g(x, y) \text{ and } g(x, y) = 2.$$

   This translates into the following equations

   $$\begin{aligned} 3 &= \lambda 2x \\ 2 &= \lambda 2y \\ x^2 + y^2 &= 2. \end{aligned}$$

   Now we can express x, y in terms of λ, and use the last equation to find the two possible values of λ.

1. Find the point on the line $y = x + 2$ which is closest to the origin. Hint: Set $f(x, y) = x^2 + y^2 = ||(x, y)||^2$ and $g(x, y) = x + 2 - y$. Explain why it is enough to find the minimum of $f(x, y)$ on the level set $g(x, y) = 0$.

   Solution: Clearly the level set $g(x, y) = x+2-y = 0$ just describes the line $y = x+2$. Finding the point closest to the origin means that we have to find the point on the level set $g(x, y) = 0$ such that $||(x, y)|| = \sqrt{x^2 + y^2}$ (which measures the distance to the origin) is minimal. Since this is a difficult function let's try to find the minimum of $||(x, y)||^2$, i.e. of $f(x, y) = x^2 + y^2$. The reason why we can do this is that any positive function h is minimal exactly when h^2 is minimal.

   So let's find the minimum of $f(x, y)$ on the level set $g(x, y) = 0$. This is clearly a case for Lagrange multipliers. So we have to find x, y, λ such that

   $$\nabla f(x, y) = \lambda \nabla g(x, y) \text{ and } g(x, y) = 2.$$

This translates into the following equations

$$\begin{aligned} 2x &= \lambda \\ 2y &= -\lambda \\ x + 2 - y &= 0. \end{aligned}$$

Now we can express x, y in terms of λ, and use the last equation to find the only possible values for λ.

11. p. 348, problems (6): Let D be the region bounded by the positive x and y–axes and the line $3x + 4y = 10$. Compute

$$\int\int_D x^2 + y^2 \, dA.$$

Solution: It's a good idea to first sketch the region D. The easiest way to do this is to see that the line $3x + 4y = 10$ is just the graph of $y = \frac{10}{4} - \frac{3}{4}x$. We now have to find the limits of integration since we want to write:

$$\int\int_D x^2 + y^2 \, dA = \int_{x=a}^{x=b} \int_{y=\phi_1(x)}^{y=\phi_2(x)} x^2 + y^2 \, dydx.$$

Since $x \geq 0, y \geq 0$ we know that the lower limit for x is zero. Furthermore x is maximal when $y = 0$. So the maximal x is $\frac{10}{3}$. So we get $a = 0, b = \frac{10}{3}$. So now we know the x–limits. Given some x between $a = 0$ and $b = \frac{10}{3}$ the possible y–values are between $y = 0$ and $y = \frac{10}{4} - \frac{3}{4}x$. So we get

$$\int\int_D x^2 + y^2 \, dA = \int_{x=0}^{x=\frac{10}{3}} \int_{y=0}^{y=\frac{10}{4}-\frac{3}{4}x} x^2 + y^2 \, dydx.$$