1. Compute \( \int_C \mathbf{F}(x,y) \cdot d\mathbf{s} \) for \( \mathbf{F}(x,y) = (x - y, y - x) \) and \( C \) the square bounded by \( x = 0, x = 1, y = 0, y = 1 \).

Solution: One could just parametrize the boundary of the square, but this involves four line segments, so it’s better to use Green’s theorem. Recall that Green’s theorem says that if \( R \) is a region in \( \mathbb{R}^2 \), then
\[
\int_{\text{boundary of } R} \mathbf{F} \cdot d\mathbf{s} = \iint_R \left( \frac{\partial}{\partial x} \text{(second component of } \mathbf{F}) - \frac{\partial}{\partial y} \text{(first component of } \mathbf{F}) \right) dA.
\]
So here we get
\[
\int_C \mathbf{F} \cdot d\mathbf{s} = \int_{x=0}^{x=1} \int_{y=0}^{y=1} -1 - (-1) dA = 0.
\]

2. Compute \( \int_C \mathbf{F}(x,y) \cdot d\mathbf{s} \) for \( \mathbf{F}(x,y) = (\tan^{-1}\left(\frac{y}{x}\right), \ln(x^2 + y^2)) \) and \( C \) the boundary of the region defined by the polar coordinate inequalities \( 1 \leq r \leq 2, 0 \leq \theta \leq \pi \).

Solution: We again use Green’s theorem, we get
\[
\int_C \mathbf{F} \cdot d\mathbf{s} = \iint_R \frac{\partial}{\partial x}(\ln(x^2 + y^2)) - \frac{\partial}{\partial y}(\tan^{-1}\left(\frac{y}{x}\right)) dA =
\int \int_R \frac{2x}{x^2 + y^2} - \frac{1}{1 + \tan^2\left(\frac{y}{x}\right) x} dA.
\]
We have to integrate over the region defined by the polar coordinate inequalities \( 1 \leq r \leq 2, 0 \leq \theta \leq \pi \). This suggests that we should switch to polar coordinates. The integral then becomes
\[
\int_{r=1}^{r=2} \int_{\theta=0}^{\theta=\pi} \left[ \frac{2r \cos(\theta)}{r^2} - \frac{1}{1 + \tan(\theta)^2} \frac{1}{x} \right] r d\theta dr.
\]
Note that we had to add an \( r \)-factor because we switched to polar coordinates. This simplifies to
\[
\int_{r=1}^{r=2} \int_{\theta=0}^{\theta=\pi} 2 \cos(\theta) - \frac{1}{\sin^2(\theta) + \cos^2(\theta)} \frac{1}{\cos(\theta)} d\theta dr.
\]
which becomes
\[
\int_{r=1}^{r=2} \int_{\theta=0}^{\theta=\pi} 2 \cos(\theta) - \cos(\theta) d\theta dr.
\]