Homework 12 – Solutions

1. p. 471, problem 7.
Solution: So let S be the surface given by $x = \frac{1}{\sqrt{y^2+z^2}}$ with $x \in [1, \infty)$. To visualize this it's perhaps easiest to think of the x–axis as looking up, then S looks like an infinite funnel, its broad at the base but gets thinner and thinner as you go up.
Let's first compute the volume. We can think of S as the graph of the function $x = \frac{1}{\sqrt{y^2+z^2}}$. The condition $x \in [1, \infty)$ translates into y, z being in the circle of radius 1 around the origin (with the origin excluded). Since we integrate over a circle we write
$$\begin{aligned} y &= r\cos(\theta) \\ z &= r\sin(\theta) \end{aligned}$$
with $r \in (0, 1], \theta \in [0, 2\pi]$. Then the area under the graph of the function $x = \frac{1}{\sqrt{y^2+z^2}}$ equals
$$\int_{r=0}^{r=1} \int_{\theta=0}^{\theta=2\pi} \frac{1}{\sqrt{r^2}} r d\theta dr = \int_{r=0}^{r=1} \int_{\theta=0}^{\theta=2\pi} 1 d\theta dr = 2\pi.$$
(the book is perhaps slightly unclear, but for the volume we might have to remove the cylinder of height one above the base circle).
Now let's turn towards the area. We need a parametrization of S. Since the shape is the graph of a function whose domain is a circle we use r, θ. We write
$$\begin{aligned} y &= r\cos(\theta) \\ z &= r\sin(\theta) \\ x &= \tfrac{1}{\sqrt{y^2+z^2}} = \tfrac{1}{r} \end{aligned}$$
where $r \in (0, 1], \theta \in [0, 2\pi]$. So we get
$$\Phi(r, \theta) = (r\cos(\theta), r\sin(\theta), \frac{1}{r}.$$
Now we can compute T_r and T_θ and we can use the area formula.

6. p. 480, problem 1.
Solution: We have to integrate xy over the four triangles which form the sides of the tetrahedron. I will do the one which is diagonal. We can describe it using the x and y variables because from $x + z = 1$ we know how to get z in terms of x and y. So we have
$$\begin{aligned} x &= x \\ y &= y \\ z &= 1 - x. \end{aligned}$$

So we get

$$\Phi(x, y) = (x, y, 1 - x).$$

The domain for the x and y–variables is the triangle given by $(0, 0), (1, 0)$ and $(0, 1)$ in the xy–plane. So we set up the integral as follows (where $f(x, y) = xy$).

$$\int\int f(x, y)\, dS = \int_{x=0}^{x=1} \int_{y=0}^{x} f(\Phi(x, y))||T_x \times T_y||dxdy.$$

(Note the x and y–limits, that's like for double integrals over a triangle as in the old days). Now we have

$$\begin{aligned} T_x &= (1, 0, -1) \\ T_y &= (0, 1, 0) \\ T_x \times T_y &= (1, 0, 1). \end{aligned}$$

So we get

$$\begin{aligned} \textstyle\int\int f(x, y)\, dS &= \textstyle\int_{x=0}^{x=1} \int_{y=0}^{x} f(\Phi(x, y))||T_x \times T_y||dxdy \\ &= \textstyle\int_{x=0}^{x=1} \int_{y=0}^{x} f(x, y, 1 - x)||(1, 0, 1)||dxdy \\ &= \textstyle\int_{x=0}^{x=1} \int_{y=0}^{x} xy\sqrt{2}dxdy. \end{aligned}$$

The rest is a straight forward calculation.