## Homework 11 – Solutions

1. Let $\mathbf{F}(x,y) = \nabla(e^{xy} + x^5 \sin(y^4 \cdot \pi))$. Let $\mathbf{c}(t) = (t^3, 1-t^2), t \in [0,1]$. Find $\int_{\mathbf{c(t)}} \mathbf{F}\, d\mathbf{s}$.
Solution: Computing $\mathbf{F}(x,y)$ would be a big mistake. Instead we know that if we integrate the gradient vector field of a function over a path we can just take the difference of the function between the end point and the starting point. Write $f(x,y) = e^{xy} + x^5 \sin(y^4 \cdot \pi)$. We get
$$\int_{\mathbf{c(t)}} \mathbf{F}\, d\mathbf{s} = \int_{\mathbf{c(t)}} \nabla f(x,y)\, d\mathbf{s} = f(\mathbf{c(1)}) - \mathbf{f(c(0))} = \mathbf{f(1,0)} - \mathbf{f(0,1)}.$$

2. Let $\mathbf{F}(x,y) = (xy, x+y)$. Let $\mathbf{c}(t) = (t,0), t \in [0,2]$ and $\mathbf{d}(t) = (t, 1-(t-1)^2), t \in [0,2]$. Compute the line integrals $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ and $\int_{\mathbf{d}} \mathbf{F} \cdot d\mathbf{s}$. Compare the results and explain why $\mathbf{F}$ is not $\nabla f$ for some function $f$.
Solution: The two path integrals give different answers (that's a straightforward computation), but $\mathbf{c}$ and $\mathbf{d}$ have the same starting point $(0,0)$ and the same end point $(2,0)$. IF $\mathbf{F}$ was a gradient vector field of a function, then starting at $(0,0)$, going along $\mathbf{c}$ to $(2,0)$ and then going back to $(0,0)$ using $\mathbf{d}$ (with the opposite direction) would give zero for the line integral. But this line integral is just the difference of the integrals along $\mathbf{c}$ and $\mathbf{d}$ (difference because we now go BACK along $\mathbf{d}$), but the difference is non–zero.

3. Let $C$ be the triangle given by $A = (0,0), B = (0,1), C = (1,1)$. View $C$ as a directed simple curve with counterclockwise orientation. Compute $\int_C \mathbf{F} \cdot d\mathbf{s}$ with $\mathbf{F}(x,y) = \nabla f(x,y)$ where $f(x,y) = e^{\sqrt{y}x} \sin(x \cos(y))$.
Solution: Since $\mathbf{F}(x,y)$ is a gradient vector field and $C$ is a closed curve we immediately get that the line integral is zero (because the line integral for a gradient vector field is the difference of the original function at the end point and the function at the starting point, but for a closed curve the end point and the starting point are the same).

6. Let $C$ be the 'infinite' cylinder of radius 2 around the $z$–axis. Now let $S$ be the surface given by cutting $C$ along the planes $z = 3$ and $z = x$. Find a parametrization of $S$.
Solution: Since $S$ is part of a cylinder we should use $\theta$ and $z$ as our variables (we don't need $r$ since it's the same along the cylinder). We get
$$\begin{aligned} x &= 2\cos(\theta) \\ y &= 2\sin(\theta) \\ z &= z. \end{aligned}$$
The limits for $\theta$ are $\theta \in [0, 2\pi]$, the limits for $z$ are $z \in [x, 3]$, which we can rewrite as $z \in ]2\cos(\theta), 3]$.

7. Let $S$ be the triangle given by the points $P = (0, 0, 1)$, $Q = (2, 3, 1)$ and the origin. Find a parametrization for $S$.
Solution: Note that $S$ is part of the plane described by $P$, $Q$ and the origin. An equation for the plane would be

$$l(0, 0, 1) + \mu(2, 3, 1)$$

where $l, \mu \in \mathbb{R}$. The possible values for $l$ are $l \in [0, 1]$ and for $\mu$ they are $\mu \in [0, l]$. Perhaps that's easiest to see if you sketch the domain for $l, \mu$ in a diagram with $l$ and $\mu$–axis and think about what $l, \mu$-values correspond to the line segment connecting $P$ and $Q$.