1. Let
$$f(x,y) = x^3 + 3x^2y^2 + y^3.$$
   (a) Give the equation for the tangent plane of $f(x,y)$ at the point $(-1,1)$.
   (b) Give the quadratic Taylor polynomial for $f(x,y)$ at the point $(-1,1)$.

Solution:

(a). A normal vector for the tangent plane is $(3,-9,1)$, so the tangent plane is given by
$$3(x+1) - 9(y-1) + (z-3) = 0.$$

Alternatively, you can get the tangent plane from the first order Taylor polynomial:
$$\begin{aligned} T_1(x,y) &= f(-1,1) + \left.\frac{\partial f}{\partial x}\right|_{(-1,1)} (x+1) + \left.\frac{\partial f}{\partial y}\right|_{(-1,1)} (y-1) \\ &= 3 + 3(x+1) - 9(y-1). \end{aligned}$$

(b) Take the second order derivatives and evaluate at $(-1,1)$.
$$\begin{aligned} \left.\frac{\partial^2 f}{\partial x^2}\right|_{(-1,1)} &= 6x + 6y^2\big|_{(-1,1)} = 0 \\ \left.\frac{\partial^2 f}{\partial y^2}\right|_{(-1,1)} &= 6x^2 + 6y\big|_{(-1,1)} = 12 \\ \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(-1,1)} &= 12xy\big|_{(-1,1)} = -12 \end{aligned}$$

So the Taylor expansion is
$$\begin{aligned} T_2(x,y) = T_1(x,y) &+ \frac{1}{2}\left.\frac{\partial^2 f}{\partial x^2}\right|_{(-1,1)} (x+1)^2 + \frac{1}{2}\left.\frac{\partial^2 f}{\partial y^2}\right|_{(-1,1)} (y+1)^2 \\ &+ \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(-1,1)} (x+1)(y-1) \\ &= T_1(x,y) + 0(x+1)^2 + 6(y-1)^2 + (-12)(x+1)(y-1) \\ &= 3 + 3(x+1) - 9(y-1) + 6(y-1)^2 - 12(x+1)(y-1). \end{aligned}$$

Problems from the book which might be helpful as well:

1. p. 255, problems 2, 4 (where 'few' means 'two')
2. p. 256, problems 9, 10, 11, 15
3. p. 257, problem 22

4. p. 365, problems 1, 3, 5, 6
5. p. 366, problem 10 (what happens if you switch the order of integration?)
6. p. 366, problem 14
7. p. 367, problems 25, 26
8. p. 417, problems 4 (a), (b), (c)
9. p. 418, problems 5, 6, 7, 10, 13, 16, 17, 19 (a)

Solutions to the above problems: