

Special Links

work in progress with Jason DeBlois & Henry Wilton

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Thanks for listening!

Introduction

Some attractive links

Construction

Properties

Special square complexes

X_n is virtually special

M_n is virtually fibered

A motivating question

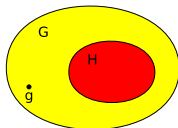
Virtually fibered: W. Thurston asked whether every hyperbolic 3-manifold has a finite cover that is a surface bundle over the circle.

Remark: A manifold (group) is **virtually** $*$ if it has a finite cover (finite index subgroup) with the property $*$.

A related question asks whether every hyperbolic 3-manifold virtually contains an essential surface. (Virtually Haken).

Subgroup Separability

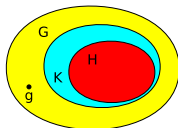
A subgroup H of a group G is **separable** if for every $g \in G \setminus H$ there is a finite index subgroup $K < G$ with $H < K$ and $g \notin K$.



Subgroup separability is often useful in lifting immersions to embeddings in finite covers.

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LERF and GFERF

- ▶ G is **LERF** (locally extended residually finite) if every finitely generated subgroup is separable.
- ▶ A hyperbolic manifold group G is **GFERF** (geometrically finite extended residually finite) if every geometrically finite group is separable.

(A subgroup H is **geometrically finite** if a neighborhood of the convex core of \mathbb{H}^n/H has finite volume.)

Commensurability

Two manifolds M and M' are **commensurable** if they share a finite cover.

Two subgroups $H_1, H_2 < G$ are commensurable if they can be conjugated so that

$$[H_i : H_1 \cap H_2] < \infty \quad \text{for } i = 1, 2.$$

- ▶ Virtual properties are well-defined on commensurability classes.
- ▶ LERF and GFERF are well-defined on commensurability classes.

Scott's Theorem (1977)

P. Scott proved that surface groups ($g \geq 2$) are LERF.

- ▶ Groups generated by reflections in a **right-angled** polygon in \mathbb{H}^2 are GFERF.
- ▶ Every surface group ($g \geq 2$) is commensurable to such a reflection group.
- ▶ Every finitely generated subgroup of a surface group ($g \geq 2$) is geometrically finite.

Higher dimensional version

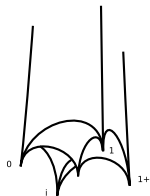
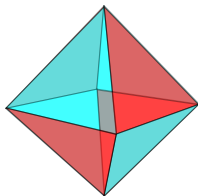
Folklore: Scott's result extends to show that groups commensurable to higher dimensional right-angled reflection groups are GFERF.

This turned out to be hard (esp. noncompact case) – Proved in 2006 by I. Agol, D. Long, A. Reid.

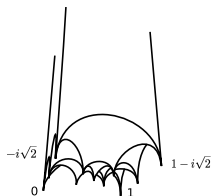
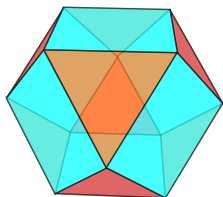
In the same paper, they also showed that many (arithmetic) 3-manifold groups **not** commensurable to reflection groups are GFERF. They show that these groups inject into higher dimensional right-angled reflection groups.

Goal:

Find examples of (nonarithmetic) hyperbolic 3-manifold groups that are GFERF that are **not** commensurable to right-angled reflection groups.

M_S , a tangle in B^3 from the all-right octahedron

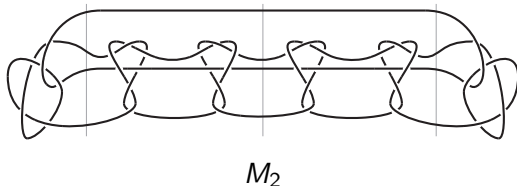
- ▶ Identify blue faces in pairs \longrightarrow M_S is hyperbolic.
- ▶ Red faces \longrightarrow Totally geodesic boundary of M_S .
- ▶ Blue faces \longrightarrow Totally geodesic surface $\perp \partial M_S$.

M_{T_0} , a tangle in $S^2 \times I$ from the all-right cube-octahedron

- ▶ Identify blue faces in pairs $\longrightarrow M_{T_0}$ is hyperbolic.
- ▶ $\left\{ \begin{array}{l} \text{Red faces} \\ \text{Orange faces} \end{array} \right\} \longrightarrow \text{Geodesic boundary of } M_{T_0}.$
- ▶ Blue faces \longrightarrow Totally geodesic surface $\perp \partial M_{T_0}$.

The hyperbolic link complement M_n

M_n is formed from M_S , $\overline{M_S}$, M_{T_0} , and $\overline{M_{T_0}}$ by attaching them with isometries between their boundaries.



Geodesic surfaces

- ▶ M_n has a series of disjoint “vertical” totally geodesic 4-punctured spheres which separate M_n into its building blocks.
- ▶ M_n contains a “horizontal” totally geodesic surface which intersects each vertical surface at right-angles.
- ▶ M_n has an ideal cell decomposition made up of all-right ideal octahedra and cube-octahedra.

Commensurability properties

Theorem (Chesebro-Deblois)

For each $n \in \mathbb{N}$, M_n is hyperbolic, has trace field $\mathbb{Q}(i, \sqrt{2})$, and is not commensurable with M_m if $m \neq n$.

- ▶ **Fact:** M_n is not arithmetic.
- ▶ **Fact:** M_n is not commensurable to any reflection group coming from a union of all-right octahedra and cube-octahedra.
- ▶ **We believe:** M_n is not commensurable to any 3-dimensional right-angled reflection group.

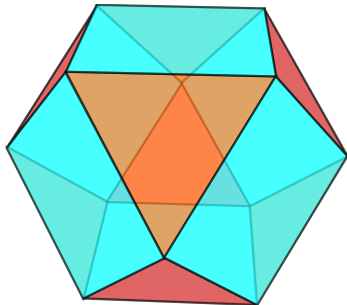
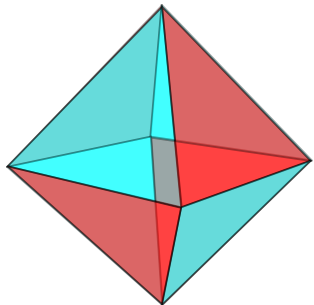
Square complexes

A **square complex** is a 2-dimensional CW-complex whose 2-cells are Euclidean squares and whose attaching maps are Euclidean isometries.

It is nonpositively curved if the induced metric is locally a CAT(0) metric space.

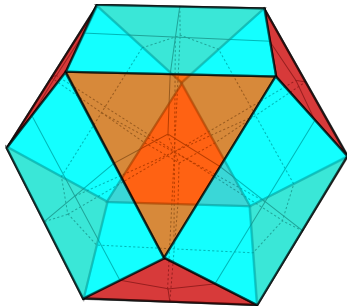
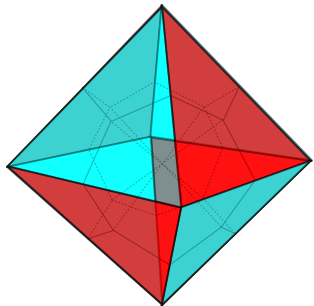
Natural square complexes for M_n

The square complex X_n for M_n is the first subdivision of the duals for the ideal octahedra and cube-octahedra.



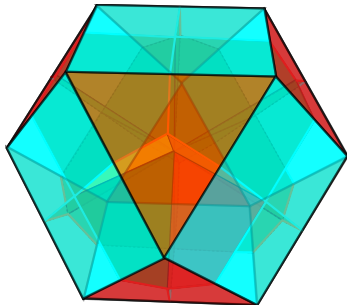
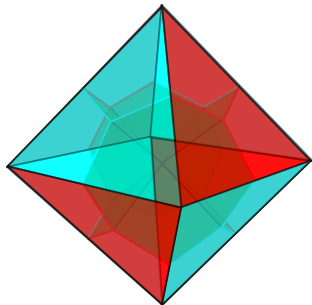
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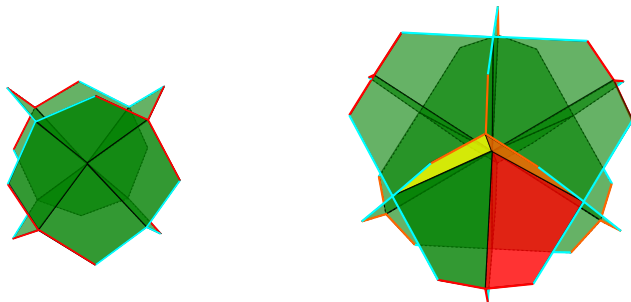
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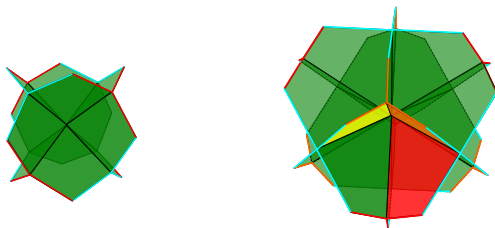
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Gromov's link condition

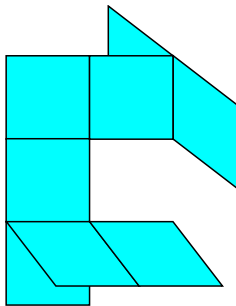
A square complex is nonpositively curved iff no vertex link contains a cycle of length < 4 .

These complexes are nonpositively curved because cubes and rhombic dodecahedra don't have any triangular faces.



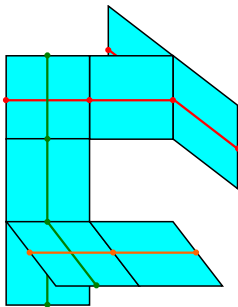
Hyperplanes

Given a square complex X form a simplicial graph Y whose edges are the midlines of X and whose vertices are the midpoints of edges of X . Each component of Y is a hyperplane of X .



Hyperplanes

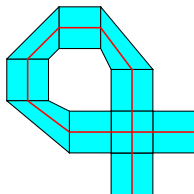
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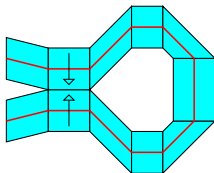
Special square complexes – F. Haglund, D. Wise

A square complex is **special** if

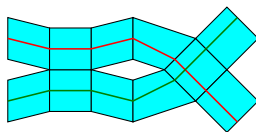
1. Every hyperplane embeds
2. No hyperplane directly self-osculates
3. No two hyperplanes inter-osculate.



Self-intersecting
hyperplane



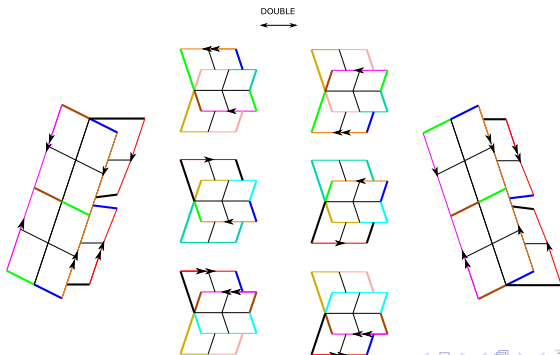
Directly self-osculating
hyperplane



Inter-osculating
hyperplanes

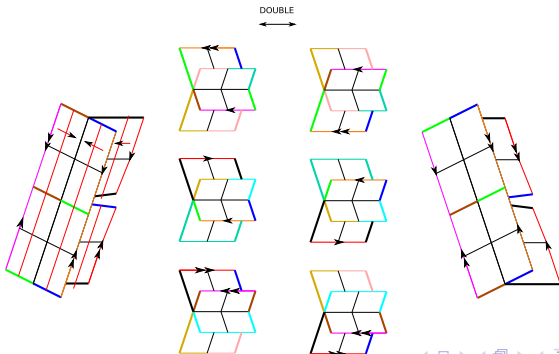
Hyperplanes in the link complex

- ▶ Hyperplanes are either horizontal or vertical.
- ▶ Not special – only direct self-osculations.
- ▶ Hyperplane subgroups are geodesic.
- ▶ Hyperplanes are horizontal or vertical. (\mathcal{VH} -complex.)



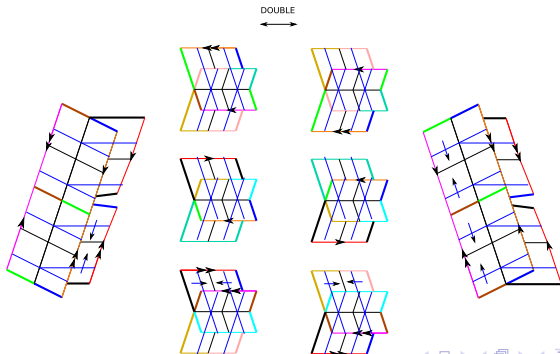
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Haglund-Wise Theorem

Theorem (F. Haglund, D. Wise)

If X is a virtually special square complex then there is a finite index subgroup of $\pi_1 X$ which is isomorphic to a subgroup of a right-angled Artin group.

Therefore, $\pi_1 X$ enjoys the same separability properties as right-angled Artin groups. (GFERF)

Use that X_n is a \mathcal{VH} -complex

X_n is a \mathcal{VH} -complex

Apply another Haglund-Wise Theorem:

\Rightarrow Need only fix vertical hyperplanes

The only pathologies in the vertical hyperplanes are direct self-osculations.

Long's Theorem

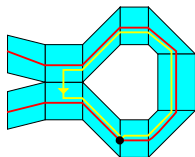
D. Long has a theorem about separability of totally geodesic surface subgroups. (The stabilizer of the limit set is separable.)

Our hyperplane subgroups are carried by embedded totally geodesic surfaces.

Long's Theorem says that the hyperplane subgroups are separable.

Hyperplane separability implies virtually special

Since the offending vertical hyperplanes have separable groups, we can pass to a finite cover where they don't directly self-osculte.



There is a finite set $\{b_1, \dots, b_t\} \subset \pi_1 X_n$ so that we need to pass to a cover so that

- ▶ no element of the form $\pi_1 H \cdot b_i \cdot \pi_1 H$ lifts
- ▶ $\pi_1 H$ lifts

X_n is virtually special

Theorem

X_n is virtually special.

Corollary

$\pi_1 M_n$ virtually embeds in a right-angled Artin group.

Corollary

$\pi_1 M_n$ is GFERF.

Recall: M_n is nonarithmetic and we don't think that M_n is commensurable to a right-angled reflection group.

Agol's definition

A group G is residually finite \mathbb{Q} -solvable (RFRS) if there is a sequence of subgroups $G = G_0 > G_1 > \dots$ such that

1. $G \triangleright G_i$ (normal)
2. $\bigcap_i G_i = \{1\}$ (co-final)
3. $[G : G_i] < \infty$ (finite index)
4. $G_{i+1} \geq \{\gamma \in G_i \mid \exists k \neq 0 \text{ so that } \gamma^k \in [G_i : G_i]\}$

(RHS of (4) is the first term in the rational derived series for G_i)

Fact: Subgroups of RFRS groups are RFRS.

Agol's Theorems

Theorem (I. Agol)

The following groups are virtually RFRS:

1. *surface groups*
2. *reflection groups*
3. *right-angled Artin groups*

Theorem (I. Agol)

Let M is a connected orientable irreducible 3-manifold with $\chi(M) = 0$. If $\pi_1 M$ is RFRS then M is virtually fibered.

Corollary

For each n , the link complement M_n is virtually fibered.

Generalization

Tentative:

Let M be a hyperbolic 3-manifold which is tiled by right-angled ideal polyhedra. Then the associated square complex is virtually special.

Consequences:

- ▶ $\pi_1 M$ is GFERF
- ▶ M is virtually fibered.

Introduction
Some attractive links
Special square complexes
 M_n is **virtually fibered**

END