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A new filtration of the Magnus kernel

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Abstract

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For a oriented genus g surface with one boundary component, S_g, the Torelli group is the group of orientation preserving homeomorphisms of S_g that induce the identity on homology. The Magnus representation of the Torelli group represents the action on F/F'' where $F = \pi_1(S_g)$ and F'' is the second term of the derived series. I show that the kernel of the Magnus representation, $Mag(S_g)$, is highly non-trivial and has a rich structure as a group. Specifically, I define an infinite filtration of $Mag(S_g)$ by subgroups, called the higher order Magnus subgroups, $M_k(S_g)$. I develop methods for generating nontrivial mapping classes in $M_k(S_g)$ for all k and $g \geq 2$. I show that for each k the quotient $M_k(S_g)/M_{k+1}(S_g)$ contains a subgroup isomorphic to a lower central series quotient of free groups $E(g-1)_k/E(g-1)_{k+1}$. Finally I show that for $g \geq 3$ the quotient $M_k(S_g)/M_{k+1}(S_g)$ surjects onto an infinite rank torsion free abelian group. To do this, I define a Johnson–type homomorphism on each higher order Magnus subgroup quotient and show it has a highly non-trivial image.

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Chapter 1

Introduction

1.1 Background

A central goal to the field of topology is to classify manifolds up to homeomorphism. For surfaces ($2-$manifolds), this classification has been achieved. For example, oriented surfaces are completely classified by their genus and number of boundary components. With this goal completed, we seek to understand the algebraic structure of the homeomorphisms between these surfaces.

Understanding homeomorphisms of surfaces is crucial to classifying 3-manifolds. Given a surface Σ, one can obtain a 3-manifold from a "mapping torus" construction, by which one uses a homeomorphism $f : \Sigma \to \Sigma$ to obtain a quotient space $\Sigma \times I/(x, 0) \sim (f(x), 1)$. Intuitively, in this construction the "ends" of $\Sigma \times I$ are glued together by the homeomorphism f. In the case where Σ has no boundary, the result is a closed 3-manifold which fibers over the circle. If Σ has boundary components b_i and f is a homeomorphism fixing the components pointwise, one can obtain a closed 3-manifold M from the mapping torus by adding the additional identification $(y,t) \sim (y,t')$ for all $y \in \partial\Sigma$, and $t, t' \in [0,1]$. In the latter case, the mapping torus $\frac{\Sigma \times I}{(x,0)\sim(f(x),1),(y,t)\sim(y,t')}$ is called an *open book decomposition* of M. Open book decompositions have been shown by Giroux and Thurston-Winkelnkemper to correspond with contact structures on closed 3-manifolds up to positive stabilization [8] [19]. These applications make homeomorphisms of surfaces, more specifically mapping class groups, an integral tool in active areas of 3-manifold topology. These homeomorphism groups are also applied broadly in geometric group theory and algebraic geometry.

Let S be a closed orientable surface of genus g with 1 boundary component (we will sometimes denote this surface by S_g when it is necessary to be precise about the genus of the surface). The *mapping class group* of S, denoted $Mod(S)$ is the group of classes of orientation preserving homeomorphisms of S which fix the boundary pointwise. Two homeomorphisms represent the the same element (called a *mapping class*) if they are isotopic maps where the isotopy also fixes the boundary pointwise. A thorough introduction to mapping class groups can be found in [5].

Dehn twists provide some simple examples of mapping classes. A Dehn twist is a self-homeomorphism of a surface achieved by cutting the surface along a simple closed curve, twisting one of the new boundary components by one full rotation, and re-gluing, as illustrated in Figure 1.1. The Lickorish twist theorem [12] states that

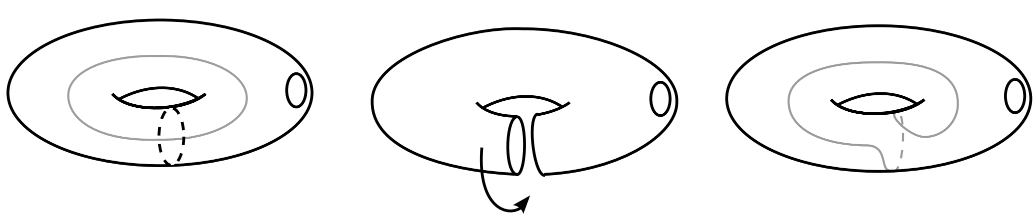

Figure 1.1: An illustration of a Dehn twist performed about the dotted curve.

the mapping class group is generated by Dehn twists.

While we will make extensive use of Dehn twists, the framework for our study is more algebraic in nature. In particular we study the mapping class group through an analysis of the fundamental group of the surface, denoted $\pi_1(S,*)$. As we restrict to maps which fix the boundary of S pointwise, by choosing a basepoint x_0 on the boundary of S, a homeomorphism $f : S \to S$ induces an automorphism $f_* : \pi_1(S,*) \to \pi_1(S,*)$. For the future, we will drop the $*$ from this notation and denote the fundamental group of S by $\pi_1(S)$ with the basepoint assumed to lie on the boundary. This correspondence induces a map

$$Mod(S) \hookrightarrow \mathrm{Aut}(\pi_1(S))$$

which is well defined on mapping classes and yields an injective homomorphism. This homomorphism provides an algebraic lens for studying the mapping class group. It is important to note that for surfaces with boundary, $\pi_1(S)$ is a free group, and hence $\mathrm{Aut}(\pi_1(S))$ is quite large. Hence to effectively employ this homomorphism we instead study mapping classes which approximate the identity automorphism. More specifically, we study subgroups of the form $\ker(Mod(S) \to \mathrm{Aut}(\pi_1(S)/H)$ where H is a characteristic subgroup of $\pi_1(S)$. There is well known commutator series $\{G_n\}$ known as the lower central series defined for any group G wherein each term G_n is a characteristic subgroup of G. Specifically, the terms of the lower central series of a group G are given inductively by $G_1 = G$, $G_k = [G_{k-1}, G]$. The mapping classes which act trivially modulo terms of the lower central series of $\pi_1(S)$ form the well–studied (for example [4], [7], [13], [14]) Johnson subgroups of the mapping class group.

More precisely, the k^{th} Johnson subgroup, $J_k(S)$, is given by $J_k(S) = \ker(Mod(S) \to \mathrm{Aut}(\pi(S)/\pi_1(S)_k)$. Since a homeomorphism which acts trivially modulo $\pi_1(S)_k$ also acts trivially modulo larger subgroups of $\pi_1(S)$, and $\pi_1(S)_n \subset \pi_1(S)_k$ for all $n > k$, the Johnson subgroups are nested and hence form a filtration of the mapping class group:

$$Mod(S) = J_1(S) \supset J_2(S) \supset \cdots \supset J_k(S) \cdots$$

The second term of this filtration, $J_2(S)$ is the subgroup of the mapping class group which acts trivially on the homology of S. This subgroup is more commonly known as the Torelli group and frequently denoted $\mathcal{I}$. The Torelli group plays a crucial role in the study of mapping class groups of surfaces as the quotient $Mod(\Sigma)/\mathcal{I}(\Sigma)$ is a well understood symplectic group.

An important tool in the study of the Torelli group is the Magnus representation. There are several Magnus representations for various groups defined via Fox calculus derivatives [2]. Of particular interest to the study of mapping class groups is the Magnus representation of the Torelli group, which can be defined as follows. Given a basis, $\{x_1, \ldots, x_n\}$, for $\pi_1(S)$, the Magnus representation of the Torelli group is map which sends a mapping class $f \in Mod(S)$ to a $2g \times 2g$ matrix with entries in $\mathbb{Z}H_1(S)$ namely,

$$f \mapsto \left(\phi\left(\frac{\partial f(x_i)}{\partial x_j}\right)\right)_{i,j}.$$

where $\frac{\partial f_*(x_i)}{\partial x_j}$ is the Fox calculus derivative of $f_*(x_i)$ with respect to x_j and $\phi : \mathbb{Z}[\pi_1(S)] \to \mathbb{Z}[H_1(S)]$ is the natural projection. However, the kernel of the Magnus representation, $Mag(S)$ also has a characterization in terms of induced automorphisms [3]. Specifically,

$$Mag(S) = \ker\left(Mod(S) \to \mathrm{Aut}(\pi_1(S)/\pi_1(S)'')\right)$$

where $\pi_1(S)'' = [[\pi_1(S), \pi_1(S)], [\pi_1(S), \pi_1(S)]]$ is the second commutator subgroup of $\pi_1(S)$.

While the Magnus representation was first introduced in the 1980s, for many years it was unknown whether the the Magnus representation was a faithful representation of the Torelli group. This remained an open question until 2001 when Suzuki constructed an explicit mapping class contained in $Mag(S_g)$ for genus $g \geq 2$ [18]. In 2009 Church and Farb proved that in fact the Magnus kernel is quite large, exhibiting infinitely many independent elements of the Magnus kernel [3]. In this paper we demonstrate that Mag_g is larger still, possessing a nontrivial filtration by subgroups, called the higher-order Magnus subgroups,

$$Mag(S_g) = M_2(S_g) \supset M_3(S_g) \supset M_4(S_g) \supset \cdots$$

for which the successive quotients are themselves infinitely generated. The previous examples of Church and Farb are all contained in $M_2(S_g) \setminus M_3(S_g)$. Hence the higher-order Magnus subgroups reveal new structure in the Magnus kernel.

1.2 Summary of results

The Johnson subgroups have provided a key tool for studying the Torelli group. While there is a clear similarity between the algebraic characterizations of the Magnus kernel and the Torelli group, attempts to define analogous tools for studying the Magnus kernel have been limited.

For any characteristic subgroup H of $\pi_1(S)$, we define an infinite family of subgroups, $J_k^H(S)$, called the higher-order Johnson subgroups. These subgroups form a filtration of the subgroup $\ker(Mod(S) \to \mathrm{Aut}(\pi_1(S)/H))$ of the mapping class group. The higher-order Johnson subgroup filtration is a generalization of the Johnson subgroup filtration of the Torelli group. In the special case where $H = [\pi_1(S), \pi_1(S)]$, we call these subgroups the higher-order Magnus subgroups, as they yield a filtration of the Magnus kernel. We show that these higher-order Johnson subgroups are have much of the natural structure known for the Johnson subgroups. These properties include the result that the higher-order Johnson subgroups are equipped with a homomorphism, analogous to the Johnson homomorphisms.

Theorem 3.1. *For each characteristic subgroup $H \subset F$ the higher-order Johnson homomorphisms,*

$$\tau_k^H : J_k^H(S) \to Hom_{\mathbb{Z}[F/H]}(H/H', H_k/H_{k+1}),$$

are well defined, group homomorphisms for $k \geq 2$.

In the special case of the Magnus subgroups, $M_k(S)$ we give an explicit way of constructing examples in $M_k(S)$ from known examples of mapping classes in $J_k(D)$ where D is a disk with n holes.

Lemma 5.1. *Let $i : D \to S$ be an embedding such that each boundary component of $i(D)$ is either separating in S, or the boundary component of S. Let $[f] \in Mod(D)$ and let f be a homeomorphism representing $[f]$. Let $f' : S \to S$ be the homeomorphism* *defined by*

$$f'(x) = \begin{cases} f(x) & x \in D \\ x & x \in S \setminus D \end{cases}$$

then if $[f] \in J_k(D)$, $[f'] \in M_k(S)$.

Using this construction, we describe an explicit subgroup of $M_k(S)/M_{k+1}(S)$ which is isomorphic to a lower central series quotient of free groups. For $E(n)$ the free group on n generators, we show the following result.

Theorem 5.6. *Let S_g be an orientable surface with genus $g \geq 3$. Then the map $\rho : E(g-1) \to Mod(S_g)$ induces a monomorphism on the quotients $\overline{\rho} : E(g-1)_k/E(g-1)_{k+1} \hookrightarrow M_k(S_g)/M_{k+1}(S_g)$ for all k.*

Finally, we construct an epimorphism onto an infinite rank torsion free abelian subgroup of $\frac{F'_k}{F'_{k+1}}$, where $F = \pi_1(S)$ is the fundamental group of S and F' is its commutator subgroup. Using Magnus homomorphism computations we prove:

Theorem 5.7. *Let S be an orientable surface with genus $g \geq 3$. Then the successive quotients of the Magnus filtration $\frac{M_k(S)}{M_{k+1}(S)}$ surject onto an infinite rank torsion free abelian subgroup of $\frac{F'_k}{F'_{k+1}}$ via the map*

$$\frac{M_k(S)}{M_{k+1}(S)} \xrightarrow{\tau'_k(-)[c_6, c_2]} \frac{F'_k}{F'_{k+1}}$$

where c_6 and c_2 are generators in the carefully chosen basis for F shown in Figure 5.8.

These results establish key tools for working with the Magnus subgroups and unveil new structure in this poorly understood subgroup of the Torelli group.

1.3 Outline of thesis

We begin in Chapter 2 by providing an overview of the original Johnson subgroups and homomorphisms. As the higher-order Johnson and Magnus subgroups and homomorphisms are a generalization of these ideas, the Johnson subgroups provide a crucial foundation for the paper.

While many of the results presented in this chapter are well known for surfaces with at most one boundary component, we also provide a detailed discussion of generalized Johnson homomorphisms on surfaces with multiple boundary components. Johnson subgroups of surfaces with multiple boundary components have been employed before, but a precise and detailed treatment of these cases have not yet appeared in the literature. We also present some new results showing some properties of traditional Johnson subgroups to apply to surfaces with multiple boundary components.

In Chapter 3 we define generalizations of the Johnson subgroups and homomorphisms called the *higher-order Johnson subgroups and homomorphisms*. A specific case of these generalized Johnson subgroups are the Magnus subgroups. These Magnus subgroups provide a filtration of the Magnus kernel and are the central focus of our study.

Chapter 4 contains some group theoretic results that are useful in proving our main theorem. These results primarily focus on the lower central series quotients of an infinitely generated free group, E, and its commutator subgroup, E'. We provide several generalizations of the basis theorem for lower central series quotients of free groups which applies to groups which are infinitely generated. We also explore the

$\mathbb{Z}[F/F']$ module structure of F'_k/F'_{k+1} where $F = \pi_1(S)$ for use in computing Magnus homomorphisms.

In Chapter 5 we prove our main results. We develop a correspondence between Magnus subgroups on surfaces with one boundary component and Johnson subgroups on disks. We use this correspondence to explore the structure and size of the successive quotients of the higher-order Magnus subgroups M_k/M_{k+1}. We demonstrate that there is a specific subgroup of $M_k(S_g)/M_{k+1}(S_g)$ that is isomorphic to the finitely generated free abelian group $E(g-1)_k/E(g-1)_{k+1}$ where $E(g-1)$ is the free group on $g-1$ generators and S_g is an oriented surface of genus g. We also show that successive quotients of the higher-order Magnus subgroups M_k/M_{k+1} are infinitely generated by displaying a surjection to a infinite rank torsion free abelian group.

Chapter 2

Johnson Subgroups and Homomorphisms

2.1 Johnson subgroups and Johnson homomorphisms for surfaces with one boundary component

Let S be an oriented surface with one boundary component. Let F denote the fundamental group of S with a basepoint chosen on the boundary of the surface (note that the fundamental group is a free group). A self–homeomorphism f of S induces an automorphism $f_* : F \to F$. This function from homeomorphisms of S to automorphisms of F is well defined on isotopy classes of homeomorphisms and yields the following monomorphism:

$$Mod(S) \hookrightarrow \mathrm{Aut}(\pi_1(S)).$$

Given a group G, the *lower central series of* G, $\{G_n\}$ is given inductively by

$G_1 = G$, $G_k = [G_{k-1}, G]$, where $[G_{k-1}, G]$ is the subgroup of G generated by elements of the form $aba^{-1}b^{-1}$, $a \in G,\ b \in G_{k-1}$.

The mapping classes which act trivially modulo terms of the lower central series of F form the well–studied Johnson subgroups of the mapping class group.

Definition 2.1. The k^{th} Johnson subgroup is the subgroup of the mapping class group given by $J_k(S) = \ker(Mod(S) \to \mathrm{Aut}(F/F_k))$.

Note that for $n > k$, as $F_n \subset F_k$ the map from $Mod(S)$ to $\mathrm{Aut}(F/F_n)$ factors through $\mathrm{Aut}(F/F_k)$:

$$\begin{tikzcd} Mod(S) \arrow[rr] \arrow[dr] & & \mathrm{Aut}(F/F_k) \\ & \mathrm{Aut}(F/F_n) \arrow[ur] & \end{tikzcd}$$

Hence $\ker(Mod(S) \to \mathrm{Aut}(F/F_n)) \subset \ker(Mod(S) \to \mathrm{Aut}(F/F_k))$ and thus $J_n(S) \subset J_k(S)$. We achieve a filtration of the Torelli group:

$$Mod(S) = J_1(S) \supset J_2(S) \supset \cdots \supset J_k(S) \cdots$$

It is an easy task to define filtrations of the Torelli group, however the filtration by Johnson subgroups has been integral in their study. The Johnson subgroup filtration earns its important place in the study of mapping class groups for the many available tools that can be employed for their study. One class of tools frequently used in exploring the Johnson subgroups is the Johnson homomorphisms. While these homomorphisms can be defined in a variety of ways, for the course of this paper we find the following definition of the Johnson subgroups to be the most convenient.

Definition 2.2. Let $[x] \in H_1(S)$ and let x be an element of the fundamental group in the homology class $[x]$. For $f \in J_k(S)$ $f(x) \equiv x \mod \pi_1(S)_k$ or equivalently $f(x)x^{-1} \in \pi_1(S)_k$. The k^{th} Johnson homomorphism

$$\tau_k : J_k(S) \to Hom(H_1(S), \pi_1(S)_k/\pi_1(S)_{k+1})$$

is given by $\tau_k(f) = ([x] \mapsto [f(x)x^{-1}])$.

While this definition provides for easy calculations, it does not provide much clarity for why such a homomorphism is well defined. For a more thorough treatment, see [10].

Remark 2.1. It is important to note that $\ker \tau_k = J_{k+1}(S)$. That $\ker \tau_k \supset J_{k+1}(S)$ can be readily seen as for $f \in J_{k+1}(S)$, $f(x)x^{-1} \in \pi_1(S)_{k+1}$, and hence $f(x)x^{-1}$ is trivial in $\pi_1(S)_k/\pi_1(S)_{k+1}$ for all $[x]$. To see that $\ker \tau_k \subset J_{k+1}(S)$, note that if $f \in \ker \tau_k$, then $f(x)x^{-1} \in F_{k+1}$ for all classes $[x]$. Thus $f(x) = x \mod F_{k+1}$ and therefore $f \in \ker(Mod(S) \to \mathrm{Aut}(F/F_k)$. Thus $f \in J_{k+1}(S)$.

This fact provides an enlightening result when performing Johnson homomorphism computations. If f is an element of $J_k(S)$ such that $\tau_k(f) \neq 0$, then $f \notin J_{k+1}(S)$. Thus computing $\tau_k(f) \neq 0$ pins the precise location of f in the Johnson filtration to $J_k(S)/J_{k+1}(S)$.

2.2 Johnson subgroups and homomorphisms for surfaces with multiple boundary components

Through the course of this paper we will employ Johnson homomorphisms on surfaces with multiple boundary components. There are many variations for Johnson subgroups with multiple boundary components. In addition, there are many cases in which surfaces with multiple boundary components are overlooked in the study of mapping class groups. Resources detailing definitions and results concerning surfaces with multiple boundary components are sparse difficult to find. Treatment of Johnson subgroups and Johnson homomorphisms for surfaces with multiple boundary components can be found in [4], [15], [16]. We will take this opportunity to address an analog of the Johnson machinery in detail for surfaces with multiple boundary components, through a perspective compatible with our following definitions of higher-order Johnson subgroups.

Let Σ be an orientable surface with $m+1$ boundary components. Choose an ordering of the boundary components $b_0, \ldots, b_m$. Let p_i be a point on the i^{th} boundary component (we choose p_0 to be the basepoint for $\pi_1(\Sigma)$). Choose arcs A_i which originate from p_0 and terminate at p_i for each $0 < i < m$.

Definition 2.3. Let $f \in Mod(\Sigma)$. Then f is in the k^{th} Johnson subgroup of Σ, $J_k(\Sigma)$ if f satisfies the following two properties:

1. For $\gamma \in \pi_1(\Sigma)$, $f_*(\gamma)\gamma^{-1} \in \pi_1(\Sigma)_k$.

2. For all A_i, $\left[f(A_i)\overline{A_i}\right] \in \pi_1(\Sigma)_k$

where $\overline{A_i}$ is the reverse of the path A_i.

Note that when $m = 0$ we obtain from this definition the standard Johnson subgroups for a surface with a single boundary component. Note also that the combination of properties (1) and (2) show that the Johnson subgroups on surfaces with multiple boundary components are independent of the ordering of the boundary components, the choices of points p_i and the choices of arcs A_i.

Given this definition of Johnson subgroups on surfaces with multiple boundary components, we would like to be able to easily generate examples of elements in the Johnson subgroups for these surfaces. Below is a generalization of a result of Morita [13], which allows us to generate examples in the Johnson subgroups via commutators.

Lemma 2.2. *Let Σ be an oriented surface with at least one boundary component. Let $f_k \in J_k(\Sigma)$ and $f_l \in J_l(\Sigma)$. Then the commutator $[f_k, f_l]$ is contained in $J_{k+l-1}(\Sigma)$.*

Proof. It suffices to prove the statement for $k \leq l$. To show that $[f_k, f_l]$ is contained in $J_{k+l-1}(\Sigma)$, we must show the following two conditions are satisfied:

(i) For each arc A_i connecting the basepoint to the i^{th} boundary component, $[f_k, f_l](A_i)\overline{A_i} \in F_{k+l-1}$.

(ii) For all $x \in \pi_1(\Sigma)$, $[f_k, f_l](x)x^{-1} \in F_{k+l-1}$.

A result of Morita ([13] Corollary 3.3) shows condition (ii) to be satisfied in the case where Σ is a closed surface with a marked point. In addition, Morita [13] shows when Σ is a closed surface with a marked point, $y \in F_l$, $f_{k*}(y)y^{-1} \in F_{k+l-1}$. While these results are not stated for surfaces with multiple boundary components, the proofs employ only the property that for $\gamma \in \pi_1(\Sigma)$ and $f \in J_n(\Sigma)$, $f_*(\gamma)\gamma^{-1} \in \pi_1(\Sigma)_n$. As this property also holds for surfaces Σ with multiple boundary components, identical arguments show analogous results for the case of multiple boundary components. We will employ these results for surfaces Σ with multiple boundary components with no further proof.

It suffices to show that $[f_k, f_l](A_i)\overline{A_i} \in F_{k+l-1}$. For this we follow the structure of the aforementioned corollary. As $f_k \in J_k(\Sigma)$, $f_k(A_i)\overline{A_i} \in F_k$. Let $x_k = f_k(A_i)\overline{A_i} \in F_k$ and note that $f_k(A_i)$ is homotopic rel endpoints to the path $x_k A_i$. Applying f_k^{-1} to this expression we find $A_i \simeq f_k^{-1}(x_k) f_k^{-1}(A_i)$ or $f_k^{-1}(\overline{x_k})A_i \simeq f_k^{-1}(A_i)$. Similarly we know $f_l \in J_l(\Sigma)$, $f_l(A_i)\overline{A_i} \in F_l$. Defining $x_l = f_l(A_i)\overline{A_i} \in F_l$ we have $f_l(A_i) \simeq x_l A_i$ and $f_l^{-1}(\overline{x_l})A_i \simeq f_l^{-1}(A_i)$. Using this we can perform the following computation:

$$
\begin{aligned}
[f_k, f_l](A_i) &= f_k f_l f_k^{-1}(f_l^{-1}(A_i)) \\
&\simeq f_k f_l f_k^{-1}(f_l^{-1}(x_l^{-1})A_i) \\
&\simeq f_k f_l (f_k^{-1} f_l^{-1}(x_l^{-1}) f_k^{-1}(A_i)) \\
&\simeq f_k f_l (f_k^{-1} f_l^{-1}(x_l^{-1}) f_k^{-1}(x_k^{-1})A_i) \\
&\simeq f_k (f_l f_k^{-1} f_l^{-1}(x_l^{-1}) f_l f_k^{-1}(x_k^{-1}) f_l(A_i)) \\
&\simeq f_k (f_l f_k^{-1} f_l^{-1}(x_l^{-1}) f_l f_k^{-1}(x_k^{-1}) x_l A_i) \\
&\simeq [f_k, f_l](x_l^{-1}) f_k f_l f_k^{-1}(x_k^{-1}) f_k(x_l) f_k(A_i) \\
&\simeq [f_k, f_l](x_l^{-1}) f_k f_l f_k^{-1}(x_k^{-1}) f_k(x_l) x_k A_i
\end{aligned}
$$

This gives us the following expression for the homotopy class of the loop $[f_k, f_l](A_i)\overline{A_i}$

in $\pi_1(\Sigma)$:

$$\left[[f_k, f_l](A_i)\overline{A_i}\right] = [f_{k*}, f_{l*}](x_l^{-1})f_{k*}f_{l*}f_{k*}^{-1}(x_k)f_{k*}(x_l)x_k$$

$$= [f_{k*}, f_{l*}](x_l^{-1})x_lx_l^{-1}f_{k*}f_{l*}f_{k*}^{-1}(x_k^{-1})x_kx_lx_l^{-1}x_k^{-1}f_{k*}(x_l)x_l^{-1}x_kx_l[x_l^{-1}, x_k^{-1}]$$

As $k \leq l$, $J_l(\Sigma) \subset J_k(\Sigma)$ and so $[f_k, f_l] \in J_k(\Sigma)$. As shown in [13], lemma 3.2 (i), for $y \in F_l$, $f_{k*}(y)y^{-1} \in F_{k+l-1}$. Thus $[f_{k*}, f_{l*}](x_l^{-1})x_l \in F_{k+l-1}$. Looking at this expression mod F_{k+l-1} we have that $[f_k, f_l](x_l^{-1})x_l = 1$. By [13] Lemma 3.2 (ii) the class of $f_kf_lf_k^{-1}(x_k^{-1})x_k$ is equal to that of $f_l(x_k^{-1})x_k$ and is thus also in F_{k+l-1} by [13] 3.2 (i). Similarly, $f_k(x_l)x_l^{-1} \in F_{k+l-1}$. As $[x_l^{-1}, x_k^{-1}] \in [F_l, F_k] \subset F_{k+l} \subset F_{k+l-1}$ this term also reduces to 1 modulo F_{k+l-1}. Since the entire expression is trivial mod F_{k+l-1} it follows that $\left[[f_k, f_l](A_i)\overline{A_i}\right] \in F_{k+l-1}$. □

It is natural to seek an analog for the Johnson homomorphisms which apply to surfaces with multiple boundary components. Let Δ be an open arc on b_0 originating at p_0. Let $\overline{\Sigma} = \partial(\Sigma \times I)\backslash(\text{int}(\Delta \times I))$. Note that $\overline{\Sigma}$ is a doubled version of the surface Σ with an added boundary component, as illustrated in Figure 2.1. Let $i : \Sigma \to \overline{\Sigma}$ be the natural embedding which sends Σ to $\Sigma \times \{0\}$. We give $\overline{\Sigma}$ an orientation that agrees with the orientation on Σ. We will define the Johnson homomorphisms on Σ via the Johnson homomorphisms on $\overline{\Sigma}$.

In order to do this, we first develop some algebraic tools to relate the homology and lower central series quotients of the fundamental groups of Σ and $\overline{\Sigma}$. These build on a result of Stallings ([17], Theorem 7.3), reproduced below. We first define an

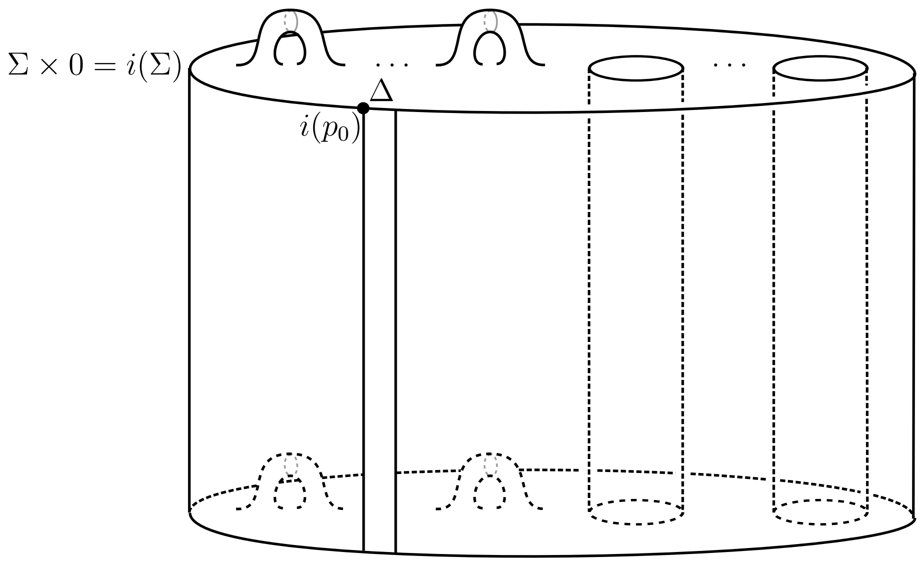

Figure 2.1: An illustration of the doubled surface $\overline{\Sigma}$.

adaptation of the lower central series: the rational lower central series. We employ this commutator series to gain insight on the lower central series of free groups.

Definition 2.4. Let G be a group. The rational lower central series of G, with terms G^r_n, is defined inductively by setting $G^r_1 = G$ and where G^r_{n+1} is the subgroup of G generated by set $S = \{[x, u] | x \in G,\ u \in G^r_n\}$ and elements w for which some power of w is a product of elements in S.

More intuitively, G^r_{n+1} is the smallest subgroup of G^r_n such that G^r_{n+1} is central in G and G/G^r_{n+1} is torsion free. Note that for a free group E, the standard lower central series quotients E/E_n are torsion free. Thus for a free group E, the lower central series of E coincides with its rational lower central series.

Theorem 2.3 (Stallings)**.** *If $f : A \to B$ a homomorphism of abelian groups inducing an isomorphism $f_* : H_1(A, \mathbb{Q}) \to H_1(B, \mathbb{Q})$, and a surjective mapping $H_1(A, \mathbb{Q}) \twoheadrightarrow$*

$H_1(B, \mathbb{Q})$. Then for all finite n, f induces isomorphisms

$$(A^r_{n-1}/A^r_n) \otimes \mathbb{Q} \cong (B^r_{n-1}/B^r_n) \otimes \mathbb{Q}$$

and for all k, $H_k(A/A^r_n) \cong H_k(B/B^r_n)$; f induces embeddings $A/A^r_n \subset B/B^r_n$ and an embedding $A/A^r_\omega \subset B/B^r_\omega$ at the first infinite ordinal ω.

We prove the following proposition employing Stalling's result.

Proposition 2.4. *Let A and B be groups with $H_2(A;\mathbb{Q}) = H_2(B;\mathbb{Q}) = 0$. Let $h : A \to B$ be a group homomorphism inducing an injection $H_1(A;\mathbb{Q}) \hookrightarrow H_1(B;\mathbb{Q})$, then for all n, h induces an injection $A/A^r_n \hookrightarrow B/B^r_n$.*

Proof. Consider the injection $h_* : H_1(A;\mathbb{Q}) \to H_1(B, \mathbb{Q})$. As $H_1(B, \mathbb{Q})$ is a $\mathbb{Q}$ vector space, it decomposes as $H_1(B;\mathbb{Q}) \cong H_1(A;\mathbb{Q}) \oplus V$ where V is a $\mathbb{Q}$ vector space. Let C be a free group of the same rank as V with generating set $\{c_i\}$ and note that $H_1(C;\mathbb{Q}) \cong V$. Let $\{e_i\}$ be a basis for V and choose elements $b_i \in B$ such that $b_i \mapsto e_i$ through the isomorphism $H_1(B;\mathbb{Q}) \cong H_1(A;\mathbb{Q}) \oplus V$. There is a unique group homomorphism $g : C \to B$ such that $c_i \mapsto b_i$. Consider the map $h * g : A * C \to B$. By construction, this is a group homomorphism which induces an isomorphism $(h * g)_* : H_1(A * C;\mathbb{Q}) \to H_1(B;\mathbb{Q})$. As $H_2(A * C) = H_2(B) = 0$, clearly the induced map $H_2(A * C;\mathbb{Q}) \to H_2(B;\mathbb{Q})$ is surjective. Hence by Stallings result, for all n, $A * C/(A * C)^r_n \overset{(h*g)_*}{\cong} B/B^r_n$. As

$$A/A^r_n \hookrightarrow A/A^r_n * C/C^r_n \cong A * C/(A * C)^r_n \overset{(h*g)_*}{\cong} B/B^r_n,$$

the map $A/A^r_n \to B/B^r_n$ induced by h is injective. $\square$

Remark 2.5. Note that for a free group E, since E is torsion free, the rational lower central series agrees with the standard lower central series, i.e. $E_n^r = E_n$. Hence for free groups A and B satisfying the conditions of Proposition 2.4 we achieve an injection $A/A_n \hookrightarrow B/B_n$ on the standard lower central series quotients. We will make extensive use of this fact throughout the paper.

For ease of notation, let us rename $C = \pi_1(\Sigma, p_0)$ and $\overline{C} = \pi_1(\overline{\Sigma}, i(p_0))$.

Lemma 2.6. *The embedding $i : \Sigma \to \overline{\Sigma}$ induces a group monomorphism*

$$\overline{i_*} : \frac{C_k}{C_{k+1}} \to \frac{\overline{C}_k}{\overline{C}_{k+1}}.$$

Proof. This is a direct application of Proposition 2.4. Note that as Σ and $\overline{\Sigma}$ are surfaces with boundary, they each deformation retract to a wedge of circles. Thus $\pi_n(\Sigma) = \pi_n(\overline{\Sigma}) = 1$ for $n > 1$. Thus Σ is a $K(C, 1)$ and $\overline{\Sigma}$ is a $K(\overline{C}, 1)$. Hence $H_2(C, \mathbb{Q}) = H_2(\Sigma, \mathbb{Q}) = 0$ and $H_2(\overline{C}, \mathbb{Q}) = H_2(\overline{\Sigma}, \mathbb{Q}) = 0$. The embedding i induces a homomorphism $C \to \overline{C}$ and a monomorphism $i_* : H_1(C; \mathbb{Q}) \to H_1(\overline{C}; \mathbb{Q})$. Thus C and $\overline{C}$ satisfy the conditions of Proposition 2.4 and hence we achieve an injective homomorphism $\overline{i_*} : \frac{C_k}{C_{k+1}} \to \frac{\overline{C}_k}{\overline{C}_{k+1}}$. $\square$

We now work to relate the homology of Σ and $\overline{\Sigma}$. Let $\Theta = \overline{\Sigma} \setminus \operatorname{int}(i(\Sigma))$ and let $j : \Theta \to \overline{\Sigma}$ be the natural inclusion map. The inclusion j yields the following long exact sequence of a pair:

$$\cdots \to H_1(\Theta) \xrightarrow{j_*} H_1(\overline{\Sigma}) \xrightarrow{\pi} H_1(\overline{\Sigma}, \Theta) \to \widetilde{H}_0(\Theta) \to \widetilde{H}_0(\overline{\Sigma}).$$

Note in particular that this exact sequence provides us with an isomorphism

$$\pi : \frac{H_1(\overline{\Sigma})}{j_*(H_1(\Theta))} \xrightarrow{\cong} H_1(\overline{\Sigma}, \Theta).$$

By excision, the inclusion $i : \Sigma \to \overline{\Sigma}$ induces an isomorphism on homology:

$$i_* : H_1(\Sigma, \partial\Sigma) \xrightarrow{\cong} H_1(\overline{\Sigma}, \Theta).$$

Hence there is an isomorphism

$$\pi^{-1} i_* : H_1(\Sigma, \partial\Sigma) \xrightarrow{\cong} \frac{H_1(\overline{\Sigma})}{j_*(H_1(\Theta))}.$$

Let $[f] \in J_k(\Sigma)$ and let f be a representative homeomorphism of $[f]$. Let $\overline{f} : \overline{\Sigma} \to \overline{\Sigma}$ be given by

$$\overline{f}(x) = \begin{cases} f(x) & \text{if } x \in \Sigma \\ & \\ x & \text{if } x \notin \Sigma \end{cases}.$$

Let $i' : Mod(\Sigma) \to Mod(\overline{\Sigma})$ be the map given by $i'([f]) = \left[\overline{f}\right]$. Note that this map is well defined since isotopic maps on Σ extend to isotopic maps on $\overline{\Sigma}$. It is naturally a homomorphism.

Let η_k be the map

$$\eta_k : Hom\left(\frac{H_1(\overline{\Sigma})}{j_*(H_1(\Theta))}, \frac{\overline{C}_k}{\overline{C}_{k+1}}\right) \to Hom\left(H_1(\Sigma, \partial\Sigma), \frac{\overline{C}_k}{\overline{C}_{k+1}}\right)$$

which is the dual of the isomorphism $\pi^{-1} i_*$.

Lemma 2.7. *Given a mapping class $[f] \in Mod(\Sigma)$, $\tau_k\left(i'([f])\right) \in Hom\left(\frac{H_1(\overline{\Sigma})}{j_*(H_1(\Theta))}, \frac{\overline{C}_k}{\overline{C}_{k+1}}\right)$. Furthermore, for $[\alpha] \in H_1(\Sigma, \partial\Sigma)$,*

$$\eta_k \tau_k i'\left([f]\right)[\alpha] \in \overline{i}\left(\frac{C_k}{C_{k+1}}\right).$$

Equivalently, we have the following sequence of maps

$$J_k(\Sigma) \xrightarrow{i'} J_k(\overline{\Sigma}) \xrightarrow{\tau_k} Hom\left(\frac{H_1(\overline{\Sigma})}{j_*(H_1(\Theta))}, \frac{\overline{C}_k}{\overline{C}_{k+1}}\right) \xrightarrow{\eta_k} Hom\left(H_1(\Sigma, \partial\Sigma), \frac{\overline{C}_k}{\overline{C}_{k+1}}\right)$$
$$\xrightarrow{\overline{i}^{-1}} Hom\left(H_1(\Sigma, \partial\Sigma), \frac{C_k}{C_{k+1}}\right).$$

Proof. To prove the first statement in the lemma we examine an element $j_*[\beta] \in H_1(\overline{\Sigma})$. The element $[\beta] \in H_1(\Theta)$ has a representative element $\beta \in \pi_1(\Theta)$. As the following diagram commutes

$$\begin{array}{ccc} \pi_1(\Theta) & \xrightarrow{j_*} & \pi_1(\Sigma) \\ \downarrow & & \downarrow \\ H_1(\Theta) & \xrightarrow{j_*} & H_1(\Sigma) \end{array}$$

we have that $j_*[\beta] \in H_1(\overline{\Sigma})$ has a representative loop β which lies entirely in Θ. Thus by definition of i', for any $[f] \in J_k(\Sigma)$, $\tau_k\left(i'\left([f]\right)\right)[\beta] = \overline{f}(\beta)\beta^{-1} = \beta\beta^{-1} = 1$. Hence $\tau_k\left(i'([f])\right) \in Hom\left(\frac{H_1(\overline{\Sigma})}{j_*(H_1(\Theta))}, \frac{\overline{C}_k}{\overline{C}_{k+1}}\right)$.

To prove that $\eta_k\tau_k i'\left([f]\right)[\alpha] \in \overline{i}\left(\frac{C_k}{C_{k+1}}\right)$ let us consider the following basis for $H_1(\Sigma)$, shown in Figure 2.2.

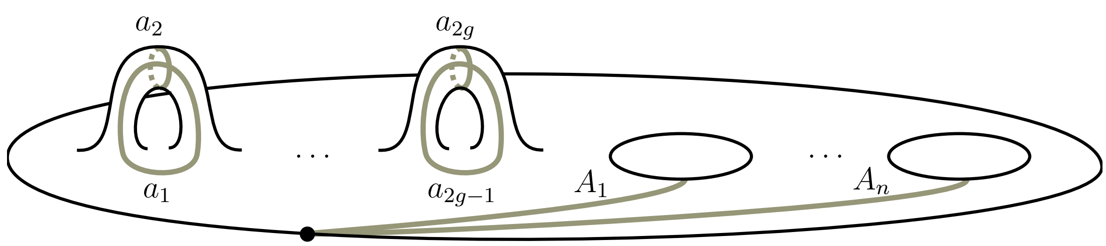

Figure 2.2: A basis for the relative homology $H_1(\Sigma, \partial\Sigma)$.

Through the map $\pi^{-1}i_*$ these basis elements map to loops in $H_1(\overline{\Sigma})$ as shown in Figure 2.3.

Note that the loops a_i include to the same homology elements of $H_1(\Sigma)$. Let the arcs A_i be parametrized by $t \in [0, 1]$. Then under this map the arcs A_i are sent to

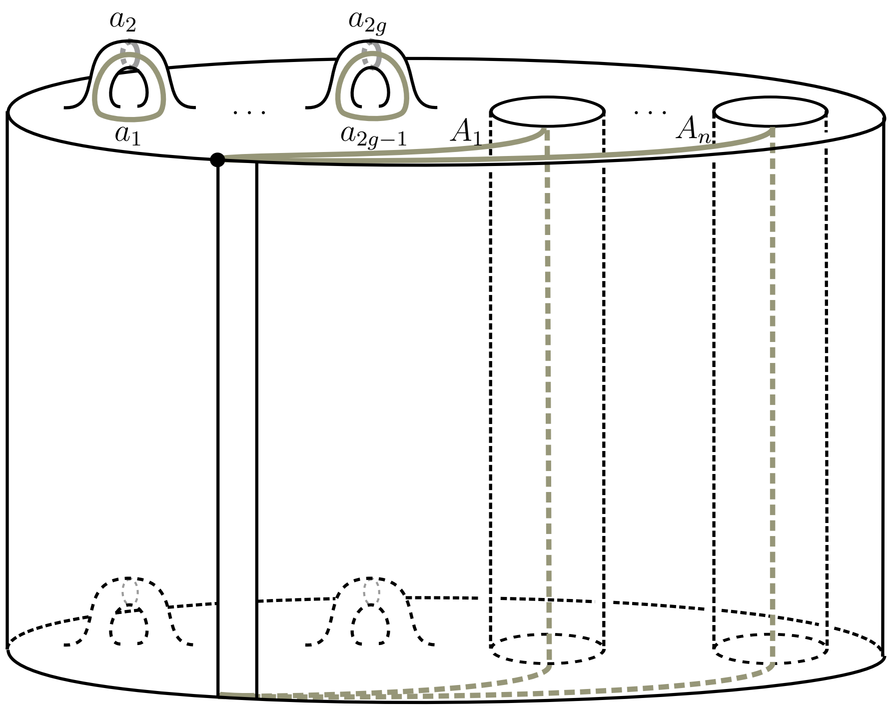

Figure 2.3: The elements of $H_1(\overline{\Sigma})$ corresponding to the basis of $H_1(\Sigma, \partial\Sigma)$ chosen in Figure 2.2.

loops c_i given by:

$$c_i(t) = \begin{cases} (A_i(4t), 0) & 0 \leq t \leq 1/4 \\ (p_i, 4t - 1) & 1/4 < t < 1/2 \\ (A_i(3 - 4t), 1) & 1/2 \leq t < 3/4 \\ (p_0, 4 - 4t) & 3/4 \leq t \leq 1 \end{cases}$$

as illustrated in Figure 2.3. First, note that for any homology class $[\alpha] \in H_1(\overline{\Sigma})$ which has a representative loop $\alpha \in i_*\pi_1(\Sigma)$ and for any $[f] \in J_k(\Sigma)$ we have

$$\begin{aligned}\tau_k(i'([f])[\alpha] &= [\overline{f}(\alpha)\alpha^{-1}] \\ &= [i_* f(\alpha)\alpha^{-1}] \\ &= \overline{i_*}[f(\alpha)\alpha^{-1}]\end{aligned}$$

Hence the image by $\tau_k(i'([f])$ of a_i yields an element of $\overline{i_*}\left(\frac{C_k}{C_{k+1}}\right)$.

Let B_i be the segment of c_i parametrized by $1/4 \leq t \leq 1$ so that $c_i = A_i \cup B_i$. Note that by definition $i'(f)$ acts by the identity on B_i and acts by f on A_i. By construction the loops c_i are based at p_0. Thus they also represent elements of $\pi_1(\overline{\Sigma})$. We will abuse notation by referring to the parametrized loop, the homotopy class, and the homology class of c_i as c_i. Then we may compute $\tau_k(i'([f])c_i$ as follows.

$$\begin{aligned}\tau_k(i'([f])c_i &= [\overline{f}(c_i)c_i^{-1}] \\ &= [\overline{f}(A_iB_i)\overline{(A_iB_i)}] \\ &= [i(f(A_i))B_i\overline{(A_iB_i)}] \\ &= [i(f(A_i))B_i\overline{B_i}\ \overline{A_i}] \\ &= [i(f(A_i))\overline{A_i}] \\ &= [i_*(f(A_i)\overline{A_i})] \\ &= \overline{i_*}[(f(A_i)\overline{A_i})]\end{aligned}$$

This shows that for each i, $\tau_k(i'([f])c_i \in \overline{i_*}\left(\frac{C_k}{C_{k+1}}\right)$. As $\tau_k(i'([f])[\alpha] \in \overline{i_*}\left(\frac{C_k}{C_{k+1}}\right)$ for all elements $[\alpha]$ of a basis for $H_1(\Sigma, \partial\Sigma)$, then for all $[\alpha] \in H_1(\Sigma, \partial\Sigma)$ we have $\tau_k(i'([f])[\alpha] \in \overline{i_*}\left(\frac{C_k}{C_{k+1}}\right)$.

This shows we have the following composition of homomorphisms:

$$J_k(\Sigma) \xrightarrow{i'} J_k(\overline{\Sigma}) \xrightarrow{\tau_k} Hom\left(\frac{H_1(\overline{\Sigma})}{j_*(H_1(\Theta))}, \frac{\overline{C}_k}{\overline{C}_{k+1}}\right) \xrightarrow{\eta_k} Hom\left(H_1(\Sigma, \partial\Sigma), \overline{i_*}\left(\frac{C_k}{C_{k+1}}\right)\right).$$

By applying $\overline{i_*}^{-1}$ on the range of $Hom\left(H_1(\Sigma, \partial\Sigma), \overline{i_*}\left(\frac{C_k}{C_{k+1}}\right)\right)$ we get the following composition:

$$J_k(\Sigma) \xrightarrow{i'} J_k(\overline{\Sigma}) \xrightarrow{\tau_k} Hom\left(\frac{H_1(\overline{\Sigma})}{j_*(H_1(\Theta))}, \frac{\overline{C}_k}{\overline{C}_{k+1}}\right) \xrightarrow{\eta_k} Hom\left(H_1(\Sigma, \partial\Sigma), \frac{\overline{C}_k}{\overline{C}_{k+1}}\right)$$
$$\xrightarrow{\overline{i_*}^{-1}} Hom\left(H_1(\Sigma, \partial\Sigma), \frac{C_k}{C_{k+1}}\right)$$

as desired. □

Definition 2.5. We define the generalized Johnson homomorphisms for surfaces with multiple boundary components $\tau_k : J_k(\Sigma) \to Hom\left(H_1(\Sigma, \partial\Sigma), \pi_1(\Sigma)_k/\pi_1(\Sigma)_{k+1}\right)$ to be the composition $\overline{i}^{-1}\eta_k\tau_k i'$ given in Lemma 2.7.

Thus to compute the Johnson homomorphism for surfaces with multiple boundary components, we must consider how the mapping class acts on all representatives of a basis for $Hom\left(H_1(\Sigma, \partial\Sigma)\right)$. In particular, this includes the action on arcs joining boundary components of Σ. It suffices to consider the action of mapping classes on arcs A_i (as described in Definition 2.3). As shown in the proof of Lemma 2.7, for these arcs we obtain the Johnson homomorphism $\tau_k([f]) = \left([A_i] \mapsto [f(A_i)\overline{A_i}]\right)$. Note that Definition 2.3 verifies that $f(A_i)\overline{A_i}$ is in fact an element of $\pi_1(\Sigma)_k$ as desired.

Chapter 3

Higher-Order Johnson Subgroups and Homomorphisms

3.1 Higher-Order Johnson Subgroups and Homomorphisms

The Johnson subgroups and homomorphisms are heavily built upon the lower central series. In this section we generalize the concepts of Johnson subgroups and homomorphisms to more general characteristic subgroups. These tools are useful in analyzing subgroups of the mapping class group which induce trivial automorphisms on F/H for any characteristic subgroup H.

Recall that S is an oriented surface with one boundary component and let $*$ be a basepoint for $\pi_1(S)$ which lies on the boundary. We are then able to define the higher-order Johnson subgroups as follows.

Definition 3.1. Let $F = \pi_1(S, *)$ and let H be a characteristic subgroup of F. Let $\phi^H : Mod(S) \to \mathrm{Aut}(F/H)$ be the map which takes a homeomorphism class in $Mod(S)$ to the induced automorphism of F/H. We define the higher-order Johnson subgroup with characteristic subgroup H, $J^H(S)$, by $J^H(S) = \ker \phi^H$. Equivalently, $J^H(S)$ is the subgroup of the mapping class group which acts trivially on F/H.

For any characteristic subgroup H of F, H_k is also a characteristic subgroup of F. The higher-order Johnson subgroup with characteristic subgroup H_k is denoted $J_k^H(S)$. As any homeomorphism acting trivially on F/H_k also acts trivially on F/H_n for $n < k$, $J_k^H(S) \subset J_n^H(S)$ for $n < k$. Hence the subgroups $J_k^H(S)$ form a filtration of $J^H(S)$: the higher-order Johnson filtration with characteristic subgroup H.

$$J^H(S) = J_1^H(S) \supset J_2^H(S) \supset J_3^H(S) \supset \cdots \supset J_k^H(S) \supset \cdots ,$$

Note that the traditional Johnson filtration is recovered by choosing $H = F$.

There is a natural structure on H_k/H_{k+1} as a left $\mathbb{Z}[F/H]$ module. Here the module action by elements $[g] \in F/H$ is given by $[g] \cdot [x] = [gxg^{-1}]$. It is clear that this action is well defined since given $g \in H$ and $x \in H_k$, the conjugate $gxg^{-1}x^{-1}$ belongs to H_{k+1}. Hence for $g \in H$, $[gxg^{-1}] = [x]$ as elements of H_k/H_{k+1}. The action by elements of $\mathbb{Z}[F/H]$ is given by the obvious extension. It is important to note that in general F/H is a nonabelian group, and hence H_k/H_{k+1} is a module over a noncommutative ring.

Having constructed subgroups analogous to the Johnson subgroups, it is natural to develop a corresponding analog to the Johnson homomorphisms.

Definition 3.2. The higher-order Johnson homomorphisms,

$$\tau_k^H(f) : J_k^H(S) \to Hom_{\mathbb{Z}[F/H]}(H/H' \to H_k/H_{k+1}),$$

are given by $\tau_k^H(f) = ([x] \mapsto [f_*(x)x^{-1}])$ where $f_* : F \to F$ is the automorphism induced by f.

Theorem 3.1. *The higher-order Johnson homomorphisms,*

$$\tau_k^H : J_k^H(S) \to Hom_{\mathbb{Z}[F/H]}(H/H', H_k/H_{k+1}),$$

are well defined, group homomorphisms for $k \geq 2$.

Proof. We will start by showing that for each $f \in J_k^H(S)$ the map $\tau_k^H(f)$ is a well defined $\mathbb{Z}[F/H]$-module homomorphism. We first show that for $[a,b] = aba^{-1}b^{-1}$, where $a, b \in H$, $\tau_k^H(f)([a,b]) = 0$ in H_k/H_{k+1}. By definition,

$$\begin{aligned}
\tau_k^H(f)([a,b]) &= f_*([a,b])[a,b]^{-1} \\
&= [f_*(a), f_*(b)][a,b]^{-1} \\
&= [ad, be][a,b]^{-1} \qquad\qquad \text{for some } d, e \in H_k.
\end{aligned}$$

Using the commutator identities $[ux, y] = {}^{u}[x,y][u,y]$ and $[x, vy] = [x,v]\,{}^{v}[x,y]$, where ${}^{h}g = hgh^{-1}$, we can simplify this further.

$$\begin{aligned}
\tau_k^H(f)([a,b]) &= {}^{a}[d, be][a, be][a,b]^{-1} \\
&= {}^{a}[d,b]\,{}^{ab}[d,e][a,b]\,{}^{b}[a,e][a,b]^{-1}
\end{aligned}$$

As $d, e \in H_k$, $[d, b], [d, e], [a, e] \in H_{k+1}$. Therefore, this expression is trivial in the quotient H_k/H_{k+1}.

We will next show that $\tau_k^H(f)$ is multiplicative. By definition, for $a, b \in H$,

$$\begin{aligned}
\tau_k^H(f)(ab) &= f_*(ab)(ab)^{-1} \\
&= f_*(a)f_*(b)b^{-1}a^{-1} \\
&= f_*(a)a^{-1}\,{}^{a}\big(f_*(b)b^{-1}\big) \\
&= f_*(a)a^{-1}f_*(b)b^{-1} && \text{as in } H_k/H_{k+1}\text{, conjugation by} \\
& && \text{an element in } H \text{ is trivial.} \\
&= \tau_k^H(f)(a)\tau_k^H(f)(b).
\end{aligned}$$

Any $w \in [H, H]$ can be written as a product of commutators $w = c_1 \cdots c_n$. This completes the proof that $\tau_k^H(f)$ is well defined, as $\tau_k^H(f)(w) = \tau_k^H(f)(c_1) \cdots \tau_k^H(f)(c_n) = 0$. This also shows that $\tau_k^H(f)$ is a group homomorphism.

To show that $\tau_k^H(f)$ is a module homomorphism for a given f we must show for $[g] \in F/H$ and $[x] \in H/H'$, $[g] \cdot \tau_k^H(f)([x]) = \tau_k^H(f)([g] \cdot [x])$. As the module action is by conjugation, we may compute as follows.

$$\begin{aligned}\tau_k^H(f)([g]\cdot[x]) &= \tau_k^H(f)\left([gxg^{-1}]\right)\\ &= \left[f_*(gxg^{-1})(gxg^{-1})^{-1}\right]\\ &= \left[f_*(g)f_*(x)f_*(g^{-1})gx^{-1}g^{-1}\right]\\ &= \left[f_*(g)g^{-1}gf_*(x)g^{-1}gf_*(g^{-1})gx^{-1}g^{-1}\right]\end{aligned}$$

This expression reduces to:

$$\tau_k^H(f)([g]\cdot[x]) = \left[(f_*(g)g^{-1})gf_*(x)g^{-1}(f_*(g)g^{-1})^{-1}gx^{-1}g^{-1}\right]$$

The element $gf_*(x)g^{-1} \in H$ as H is a characteristic subgroup. As $f \in J_k^H$, f_* acts trivially mod H_k, and thus $f_*(g)g^{-1} \in H_k$. Since $\tau_k^H(f)\left([x]\cdot[f]\right) \in H_k/H_{k+1}$, the conjugation of an element of H by an element of H_k is a trivial conjugation. This observation yields the following expression.

$$\begin{aligned}\tau_k^H(f)([x]\cdot[g]) &= \left[gf_*(x)g^{-1}gx^{-1}g^{-1}\right]\\ &= \left[gf_*(x)x^{-1}g^{-1}\right]\\ &= [g]\cdot\tau_k^H(f)(x)\end{aligned}$$

This concludes the proof that $\tau_k^H(f)$ is an $\mathbb{Z}[F/H]$-module homomorphism. It remains to show that $\tau_k^H : J_k^H(S) \to Hom_{\mathbb{Z}[F/H]}(H/H', H_k/H_{k+1})$ is a group homomorphism.

Let $f^1, f^2 \in J_k^H(S)$ and let $x \in H/H'$. We consider the image of their product by the map τ_k^H in the computation below.

$$
\begin{aligned}
\tau_k^H(f^1f^2)(x) &= (f^1f^2)_*(x)x^{-1} \\
&= f_*^1 f_*^2(x)x^{-1} \\
&= f_*^1(f_*^2(x))x^{-1} \\
&= f_*^1(f_*^2(x))(f_*^2(x))^{-1}f_*^2(x)x^{-1} \\
&= \tau_k^H(f^1)(f_*^2(x))\,\tau_k^H(f^2)(x)
\end{aligned}
$$

As $f^2 \in J_k^H(S)$, $f_*^2(x) = x$ as an element of H/H' for $k \geq 2$. Hence $\tau_k^H(f^1)(f_*^2(x)) = \tau_k^H(f^1)(x)$. Combining this with the above computation gives us the desired result: $\tau_k^H(f^1f^2)(x) = \tau_k^H(f^1)(x)\,\tau_k^H(f^2)(x)$. Thus $\tau_k^H : J_k^H(S) \to Hom_{\mathbb{Z}[F/H]}(H/H', H_k/H_{k+1})$ is a group homomorphism. □

Proposition 3.2. $J_{k+1}^H \subset \ker \tau_k^H$. *Thus*

$$\tau_k^H : \frac{J_k^H}{J_{k+1}^H} \to Hom_{\mathbb{Z}[F/H]}(H/H', H_k/H_{k+1})$$

is a well defined map.

Proof. By definition, $J_{k+1}^H = \ker\left(Mod(S) \to \mathrm{Aut}(F/H_{k+1})\right)$. Thus for $[f] \in J_{k+1}^H$, $[x] \in F/H$, we have for any representative homeomorphism $f \in [f]$, $f_*(x) = x \mod H_{k+1}$. Rewriting this expression we see $f_*(x)x^{-1} \in H_{k+1}$. Thus $[f_*(x)x^{-1}] = 1$ as an element of H_k/H_{k+1}. Hence $[f] \in \ker \tau_k'$. □

3.2 higher-order Magnus Subgroups

While the higher-order Johnson subgroups and homomorphisms are defined for any characteristic subgroup H, this machinery is of particular interest in the case where

H is the commutator subgroup of F, denoted by $[F,F]$ or F'. Through the remainder of the paper, we focus primarily on this case. For clarity, we repeat the definitions of the higher-order Johnson subgroups and homomorphisms here for this special case.

Definition 3.3. For $k \geq 2$, the higher-order Magnus subgroups $M_k(S)$ are given by $M_k(S) = J_k^{[F,F]}(S)$. Equivalently,

$$M_k(S) = \ker\left(Mod(S) \to \mathrm{Aut}(F/F'_k)\right)$$

It is of particular importance that $M_1(S) = \ker(Mod(S) \to \mathrm{Aut}(F/F^{(1)})$ is the Torelli group, and $M_2(S) = \ker(Mod(S) \to \mathrm{Aut}(F/F^{(2)})$ is the kernel of the Magnus representation of the Torelli group. Thus the higher-order Magnus filtration,

$$Mag(S) = M_2(S) \supset M_3(S) \supset \cdots \supset M_k(S) \supset \cdots$$

is a filtration of the Magnus kernel.

To investigate the structure of these higher-order Magnus subgroups, we will make frequent use of their corresponding higher-order Johnson homomorphisms.

Definition 3.4. The higher-order Magnus homomorphisms,

$$\tau'_k(f) : M_k(S) \to Hom_{\mathbb{Z}[F/F']}(F'/F'' \to F'_k/F'_{k+1}),$$

are the higher-order Johnson homomorphisms with characteristic subgroup F'.

Remark 3.3. Note that as a special case of Proposition 3.2 we have that $M_{k+1} \subset \ker \tau'_k$. Hence the Magnus homomorphisms are well defined on successive quotients

$$\tau'_k : \frac{M_k}{M_{k+1}} \to Hom_{\mathbb{Z}[F/F']}(F'/F'', F'_k/F'_{k+1}).$$

Thus, just as with the Johnson homomorphisms, for $f \in M_k$, computing $\tau'_k(f) \neq 0$ pins the location of f in the higher-order Magnus filtration precisely.

Chapter 4

Algebraic Tools

We take this opportunity to prove some algebraic results that will be of use in proving our main theorems.

4.1 Basis theorems and properties of lower central series quotients

We will make extensive use of several variations on the basis theorem for lower central series quotients of free groups [9] Theorem 11.2.4. We begin by discussing the notation and results for Hall's basis theorem before proceeding to generalizations of this result.

Let E be a free group on a free basis $x_1, \dots x_r$. We define basic commutators and construct an ordering on the basic commutators inductively as follows:

- The basic commutators of weight 1 are the generators $x_1, \dots x_r$ with $x_i < x_j$ for $i < j$.

- The basic commutators of weight n ate the commutators $c = [c_i, c_j]$ where c_i and c_j are basic commutators with weights summing to n. These are ordered lexicographically: if $c = [c_i, c_j]$ and $c' = [c'_i, c'_j]$ then $c > c'$ if $c_i > c'_i$ or if $c_i = c'_i$ and $c_j > c'_j$.

By imposing the additional requirement that $c_i < c_j$ if the weight of c_i is less than the weight of c_j, we achieve an ordering of all basic commutators.

Above we have given a precise construction of a strict ordering on basic commutators. While this ordering is consistent with Hall's original definition of ordering on basic commutators, he only insisted that the ordering be consistent with the partial ordering given by the weights and allowed for arbitrary ordering of commutators of the same weight. For our generalizations of the basis theorem, we find it advantageous to work with the specific ordering given above. This specific ordering of basic commutators has appeared before in [20].

To speak precisely about commutators and lower central series, we also introduce some new terminology.

Definition 4.1. Let $a_1, \dots, a_n \in G$. We define an n-bracketing of $a_1, \dots, a_n$ inductively by

- The 1-bracketing of a_1 is a_1
- A n-bracketing of $a_1, \dots, a_n$ is any commutator $[c_k, c_{n-k}]$ where c_k is a k-bracketing of $a_1, \dots, a_k$ and c_{n-k} is an $(n-k)$-bracketing of $a_{k+1}, \dots, a_n$.

We call an element a_i in an n-bracketing of $a_1, \dots, a_n$ an entry.

Note that the definition of n-bracketing is not very restrictive. For example, the commutators $[[a_1, a_2], [a_3, a_4]]$, $[[[a_1, a_2], a_3], a_4]$, and $[a_1, [a_2, [a_3, a_4]]]$ are all 4-bracketings of the entries a_1, a_2, a_3, a_4. Note that by the definition, any n-bracketing is an element of G_n, but it is possible for an n-bracketing to lie in a deeper term of the lower central series.

Given these definitions for commutators, we are now equipped to approach the basis theorem.

Theorem 4.1 (Hall). *If E is the free group with free generators $x_1, \ldots, x_r$ and if in a sequence of basic commutators $c_1, \ldots, c_t$ are those of weight $1, 2, \ldots, k$ then an arbitrary element g of E has a unique representation*

$$g = c_1^{e_1} \cdots c_t^{e_t} \mod E_{k+1}$$

The basic commutators of weight k form a basis for the free abelian group E_k/E_{k+1}.

Hall's basis theorem applies only to lower central series quotients of finitely generated free groups. Below, we generalize the basis theorem to hold for lower central series quotients of infinitely generated free groups.

Corollary 4.2. *If E is the free group with free generators $x_1, x_2, \ldots$ and if in a sequence of basic commutators $\{c_i\}$ are those of weight $1, 2, \ldots, k$ then an arbitrary element g of E has a unique representation*

$$g = \prod c_i^{e_i} \mod E_{k+1}$$

where $e_i = 0$ for all but finitely many i. The basic commutators of weight k form a basis for the free abelian group E_k/E_{k+1}.

Proof. Let $E(i)$ be the free group on $x_1, \ldots, x_i$. Note that the natural inclusion $E(i) \to E(j)$ for $j > i$ sends basic commutators to basic commutators and respects the ordering on basic commutators.

We first show that show that the map $\iota_{i,j} : E(i) \to E(j)$ induces an injection on the lower central series quotients $\bar{\iota}_{i,j} : E(i)_k/E(i)_{k+1} \hookrightarrow E(j)_k/E(j)_{k+1}$. Note that $H_2(E(i), \mathbb{Q}) = H_2(E(i), \mathbb{Q}) = 0$, as the wedge of i or j circles is a $K(G, 1)$ for $E(i)$ or $E(j)$ respectively. Furthermore, $\iota_{i,j}$ induces an injection $H_1(E(i), \mathbb{Q}) \to H_1(E(j), \mathbb{Q})$. By Proposition 2.4, $\iota_{i,j}$ induces an injection $\bar{\iota}_{i,j} : E(i)_k/E(i)_{k+1} \hookrightarrow E(j)_k/E(j)_{k+1}$ since free groups have the same rational lower central series and lower central series.

Let $\iota_i : E(i) \to E$ be the natural inclusion map sending $x_k \mapsto x_k$ for $k \leq i$. By an analogous argument, ι_i induces an injection $\bar{\iota}_i : E(i)_k/E(i)_{k+1} \hookrightarrow E_k/E_{k+1}$.

Given this it is easily checked that $\{E(i)_k/E(i)_{k+1}, \iota_{ij}\}$ is a directed system of groups. We will show that E_k/E_{k+1} is the direct limit of this system. For this it suffices to show that for a group G and maps $f_i : E(i)_k/E(i)_{k+1} \to G$ such that $f_i = f_j \circ \iota_{ij}$, there exists a map $f : E_k/E_{k+1} \to G$ such that $f \circ \iota_i = f_i$. For any element $x \in E_k/E_{k+1}$, x can be written as a finite length word in the generators $x_1, \ldots$. Hence $x \in E(i)_k/E(i)_{k+1}$ for some i. Define $f(x) = f_i(x)$. It is clear the resulting map f is well defined and has the desired properties.

To prove the first statement of the theorem, let $g \in E$, then $g \in E(i)$ for some i. Hence in $E(i)$ there is a unique representation for g as $g = c_1^{e_1} \cdots c_t^{e_t} \mod E(i)_{k+1}$. As $E(i)_k \subset E_k$, $g = c_1^{e_1} \cdots c_t^{e_t} \mod E_{k+1}$ is a representation of g in the desired form in E. Suppose there is another representation of this form, $g = d_1^{\epsilon_1} \cdots d_s^{\epsilon_s} \mod E_{k+1}$. There exists a j such that all of the basic commutators $c_1, \ldots, c_t, d_1, \ldots, d_s \in E(j)$.

Then by the basis theorem for finitely generated free groups these representations must be the same.

To prove the second statement, note that as E_k/E_{k+1} is a direct limit of $E(i)_k/E(i)_{k+1}$, it follows from the basis theorem for finitely generated free groups that the basic commutators of weight k generate E_k/E_{k+1}. Furthermore, for any finite collection of basic commutators of weight k $c_1, \ldots, c_m$, there is some i such that $c_1, \ldots, c_m \in E(i)_k/E(i)_{k+1}$. Hence all commutators of weight k are independent. Therefore the basic commutators of weight k form a basis for E_k/E_{k+1}. □

Lemma 4.2. *Let G be a group and let $a \in G_k$. By the definition of the lower central series, a is some n-bracketing of $a_1, \ldots a_n$ where $a_i \in G$, $n \leq k$, and $a_i \in G_{k_i}$ where $\sum_{i=1}^n k_i = k$. Let $c_1, \ldots, c_n$ be elements of G with $c_i \in G_{k_i+1}$. Let a' be the commutator a where each entry a_i is replaced by $c_i a_i$. Then $a' = ca$ for some $c \in G_{k+1}$*

Proof. We prove this using strong induction. For the case $k = 2$, $a = [a_{i_1}, a_{i_2}]$. Using the commutator identities $[ux, y] = {}^u[x, y][u, y]$ and $[x, vy] = [x, v]\, {}^v[x, y]$ we can perform the following computation:

$$\begin{aligned}
a' &= [c_1 a_{i_1}, c_2 a_{i_2}] \\
&= {}^{c_1}[a_{i_1}, c_2 a_{i_2}][c_1, c_2 a_{i_2}] \\
&= {}^{c_1}[a_{i_1}, c_2]\ {}^{c_2 c_1}[a_{i_1}, a_{i_2}][c_1, c_2 a_{i_2}] \\
&= {}^{c_1}[x_{i_1}, c_2] c_2 c_1 [a_{i_1}, a_{i_2}] c_1^{-1} c_2^{-1} [c_1, c_2 a_{i_2}] \\
&= {}^{c_1}[a_{i_1}, c_2] c_2 c_1\ {}^{[a_{i_1}, a_{i_2}]}\left(c_1^{-1} c_2^{-1} [c_1, c_2 x_{i_2}]\right) [a_{i_1}, a_{i_2}]
\end{aligned}$$

Hence for $c = {}^{c_1}[a_{i_1}, c_2]c_2c_1\ {}^{[a_{i_1},a_{i_2}]}\left(c_1^{-1}c_2^{-1}[c_1, c_2a_{i_2}]\right)$ our base case holds.

For the inductive step, suppose $a = [a_m, a_n]$ where $a_m \in G_m$, and $a_n \in G_n$. By the inductive hypothesis, $a'_m = c_m a_m$ where $c_m \in G_{m+1}$ and $a'_n = c_n a_n$ where $c_n \in G_{n+1}$

$$\begin{aligned}
a' &= [a'_m, a'_n] \\
&= [c_m a_m, c_n a_n] \\
&= {}^{c_m}[a_m, c_n a_n][c_m, c_n a_n] \\
&= {}^{c_m}[a_m, c_n]\ {}^{c_n c_m}[a_m, a_n][c_m, c_n a_n] \\
&= {}^{c_m}[a_m, c_n]c_n c_m[a_m, a_n]c_m^{-1}c_n^{-1}[c_m, c_n a_n] \\
&= {}^{c_m}[a_m, c_n]c_n c_m\ {}^{[a_m,a_n]}\left(c_m^{-1}c_n^{-1}[c_m, c_n a_n]\right)[a_m, a_n]
\end{aligned}$$

To finish the proof we must show that ${}^{c_m}[a_m, c_n]c_n c_m\ {}^{[a_m,a_n]}(c_m^{-1}c_n^{-1}[c_m, c_n a_n])$ is an element of G_{n+m+1}. Note that $[a_m, c_n] \in G_{m+n+1}$, and so any conjugate is also in E_{m+n+1}. We simplify the above expression as modulo G_{m+n+1} as follows:

$$\begin{aligned}
{}^{c_m}[a_m, c_n]c_n c_m\ {}^{[a_m,a_n]}\left(c_m^{-1}c_n^{-1}[c_m, c_n a_n]\right) &= c_n c_m\ {}^{[a_m,a_n]}\left(c_m^{-1}c_n^{-1}c_m c_n a_n c_m^{-1}a_n^{-1}c_n^{-1}\right) \\
&= c_n c_m\ {}^{[a_m,a_n]}\left([c_m^{-1}, c_n^{-1}]a_n c_m^{-1}a_n^{-1}c_n^{-1}\right) \\
&= c_n c_m\ {}^{[a_m,a_n]}\left([c_m^{-1}, c_n^{-1}][a_n c_m^{-1}](c_n c_m)^{-1}\right) \\
&= c_n c_m[a_m, a_n][c_m^{-1}, c_n^{-1}][a_n c_m^{-1}](c_n c_m)^{-1}[a_m, a_n]^{-1}
\end{aligned}$$

As $[a_m, a_n] \in G_{m+n}$, it is in the center of G/G_{m+n+1}. Also note that the commutators $[c_m^{-1}, c_n^{-1}]$ and $[a_n, c_m^{-1}]$ are elements of G_{n+m+1}. Thus modulo G_{n+m+1} the expression reduces further to:

$$\begin{aligned}{}^{c_m}[a_m, c_n]c_nc_m\ {}^{[a_m,a_n]}\left(c_m^{-1}c_n^{-1}[c_m, c_na_n]\right) &= c_nc_m[a_m, a_n][c_m^{-1}, c_n^{-1}][a_nc_m^{-1}](c_nc_m)^{-1}[a_m, a_n]^{-1} \\ &= c_nc_m[a_m, a_n](c_nc_m)^{-1}[a_m, a_n]^{-1} \\ &= 1\end{aligned}$$

Hence for $c = {}^{c_m}[a_m, c_n]c_nc_m\ {}^{[a_m,a_n]}(c_m^{-1}c_n^{-1}[c_m, c_na_n])$ our induction holds. □

Corollary 4.3. *Commutators in $\frac{G_k}{G_{k+1}}$ are linear in each entry. In other words, given $a, b, c \in G$. If $C \in \frac{G_k}{G_{k+1}}$ be an n-bracketing with $[ab, c]$ as an entry. Let C' be the commutator obtained by replacing the entry $[ab, c]$ with $[a, c][b, c]$. Then $C = C'$. In addition, if $C \in \frac{G_k}{G_{k+1}}$ with an entry $[a, bc]$ and C' is the commutator obtained by replacing the entry $[a, bc]$ with $[a, b][a, c]$, then $C = C'$.*

Proof. We will prove the first identity. The proof of the second is analogous.

In any group we have the identity

$$\begin{aligned}[ab, c] &= {}^{a}[b, c][a, c] \\ &= [a, c]\ {}^{[c,a]a}[b, c] \\ &= [a, c][\ {}^{[c,a]a}b,\ {}^{[c,a]a}c].\end{aligned}$$

Let $b \in G_{k_b}$ and $c \in G_{k_c}$. Note that the elements b and ${}^{[c,a]a}b$ share the same class in G/G_{k_b+1}. Similarly, the elements c and ${}^{[c,a]a}c$ share the same class in G/G_{k_c+1}. The result follows immediately from Lemma 4.2. □

Corollary 4.4. *Let E be the free group with free generators $x_1, x_2, \ldots$. Let $\bar{x_1}, \bar{x_2}, \ldots$ be the classes of $x_1, x_2, \ldots$ in E/E'. Consider the basic commutators $\bar{c_i}$ of weights*

$1, 2, \ldots, k$ in elements $\bar{x}_1, \bar{x}_2, \ldots$ defined in the same fashion as before (inductively from the ordering $\bar{x}_1 < \bar{x}_2 < \ldots$). An arbitrary element g of E has a unique representation

$$g = \bar{c_1}^{e_1} \cdots \bar{c_t}^{e_t} \mod E_{k+1}$$

The basic commutators of weight k form a basis for the free abelian group E_k/E_{k+1}.

Proof. By Lemma 4.2, the representation $g = {c_1}^{e_1} \cdots {c_t}^{e_t} \mod E_{k+1}$ is unchanged by sending x_i to another element in the same homology class. It follows that $g = \bar{c_1}^{e_1} \cdots \bar{c_t}^{e_t} \mod E_{k+1}$ is a well defined representation of g. The remaining statements follow directly from Corollary 4.2.

□

The following proposition establishes a relationship between bases for the lower central series quotients $\frac{E(n)_k}{E(n)_{k+1}}$ and $\frac{E(n-1)_k}{E(n-1)_{k+1}}$.

Proposition 4.3. *Let $E(n-1)$ be the free group on $\{x_1, \ldots, x_{n-1}\}$ and let $E(n)$ be the free group on $\{x_1, \ldots, x_n\}$. Let $\pi : E(n) \to E(n-1) \cong E(n)/\langle x_n \rangle$ be the natural quotient map. The kernel of the induced map*

$$\overline{\pi} : \frac{E(n)_k}{E(n)_{k+1}} \to \frac{E(n-1)_k}{E(n-1)_{k+1}}$$

is generated by weight k basic commutators which have x_n as an entry.

Proof. Let K be the subgroup of $\frac{E(n)_k}{E(n)_{k+1}}$ generated by basic commutators which have x_n as an entry. We show $K = \ker \overline{\pi}$. First, consider a basic commutator c of weight k which has x_n as an entry. We show that $\pi(c) = 1$ using strong induction.

In the first case we consider a weight 2 basic commutator. Suppose $c = [x_i, x_n]$ or $c = [x_n, x_i]$ for some x_i. Then

$$\begin{aligned}\pi(c) &= [\pi(x_i), \pi(x_n)] \\ &= [x_i, \pi(x_n)] \\ &= 1.\end{aligned}$$

and similarly for $c = [x_n, x_i]$

Suppose for induction that for all commutators c of weight less than k, that $\pi(c) = 1$. Let c be a weight k basic commutator with x_n as an entry. Then $c = [c_1, c_2]$ where c_1 and c_2 are basic commutators of weight $\leq k-1$. Either c_1 or c_2 must have x_n as an entry. If c_1 has x_n as an entry then

$$\begin{aligned}\pi(c) &= [\pi(c_1), \pi(c_2)] \\ &= [1, \pi(c_2)] \\ &= 1\end{aligned}$$

and similarly if c_2 has x_n as an entry. Thus $K \subset \ker \overline{\pi}$.

To show the opposite inclusion, let $\iota : E(n-1) \to E(n)$ be the natural inclusion map. This map induces a monomorphism $\overline{\iota} : \frac{E(n-1)_k}{E(n-1)_{k+1}} \to \frac{E(n)_k}{E(n)_{k+1}}$. Furthermore as $\overline{\pi} \circ \overline{\iota}$ is the identity map, $\overline{\pi}$ is a retract.

$$\frac{E(n-1)_k}{E(n-1)_{k+1}} \xrightarrow{\overline{\iota}} \frac{E(n)_k}{E(n)_{k+1}} \xrightarrow{\overline{\pi}} \frac{E(n-1)_k}{E(n-1)_{k+1}}$$

Suppose $c \in \frac{E(n)_k}{E(n)_{k+1}}$ and $c \notin K$. Then by the basis theorem c can be written as a product of basic commutators $c_1, \ldots, c_m$ in the generators $\{x_1, \ldots, x_n\}$. In this product there is some set of basic commutators $\{c_i | i \in A\}$ for which $c_i \notin K$. Then by definition of K, for each $i \in A$, c_i is a basic commutator in the generators $\{x_1, \ldots, x_{n-1}\}$. Thus for each $i \in A$, $c_i \in \operatorname{im} \bar{\iota}$ and the product of these basic commutators $\prod_{i \in A} c_i$ is nontrivial. Then as $\overline{\pi}(c_i) = c_i$ for each $i \in A$ and $\overline{\pi}(c_i) = 1$ for $i \notin A$, it follows that $\overline{\pi}(c) = \prod_{i \in A} c_i$. Hence $c \notin \ker \overline{\pi}$. Therefore, $K = \ker \overline{\pi}$, as desired. □

4.2 Module structure of the lower central series quotients

The first result in this section provides a homology basis for the commutator subgroup of a free group E'. An equivalent set was shown to be a basis in [1]. We provide an original proof here for completeness.

Lemma 4.4. *Let $E = E(x_1, \ldots, x_n)$ be a free group on $\{x_1, \ldots, x_n\}$. Let $B = \{ {}^{w_{i,j}}[x_i, x_j] | i < j, w_{i,j} \in Im(H_1(E(x_1, \ldots, x_j)) \hookrightarrow H_1(E))\}$. Then B is a basis for $H_1(E', \mathbb{Z})$.*

Proof. To show B is a basis for $H_1(E)$ we must show that B generates the homology of E, and also that there are no relations among the elements of B. We first show that B generates.

The set $B' = \{[x_i, x_j]^w | i < j, w \in H_1(E)\}$ is a clear generating set for $H_1(E')$. Hence to show that B is a generating set it suffices to show that any element of B' can be written as a linear combination of elements of B. Any element w of $H_1(E)$ can be written as a product $w = x_1^{m_1} \cdots x_n^{m_n}$, and hence a general element b' of B'

takes the form $b' = {}^{x_1^{m_1}\cdots x_n^{m_n}}[x_i, x_j]$. Note that if $m_l = 0$ for all l with $j < l \leq n$, then $b' \in B$. For $k > j$ we may express ${}^{x_k^{\pm 1}}[x_i, x_j]$ as follows:

$$\begin{aligned} {}^{x_k}[x_i, x_j] &= [x_i, x_j] + [x_j, x_k] - {}^{x_i}[x_j, x_k] - [x_i, x_k] + {}^{x_j}[x_i, x_k] \\ {}^{x_k^{-1}}[x_i, x_j] &= [x_i, x_j] + {}^{x_k^{-1}}[x_j, x_k] + {}^{x_k^{-1}x_i}[x_j, x_k] + {}^{x_k^{-1}}[x_i, x_k] + {}^{x_k^{-1}x_j}[x_i, x_k]. \end{aligned}$$

Note that both expressions above are linear combinations of elements of B. Given the above, we can express ${}^{vx_k^{\pm 1}}[x_i, x_j]$ as follows:

$$\begin{aligned} {}^{vx_k}[x_i, x_j] &= {}^{v}[x_i, x_j] + {}^{v}[x_j, x_k] + {}^{vx_i}[x_j, x_k] + {}^{v}[x_i, x_k] + {}^{vx_j}[x_i, x_k] \\ {}^{vx_k^{-1}}[x_i, x_j] &= {}^{v}[x_i, x_j] + {}^{vx_k^{-1}}[x_j, x_k] + {}^{vx_k^{-1}x_i}[x_j, x_k] + {}^{vx_k^{-1}}[x_i, x_k] + {}^{vx_k^{-1}x_j}[x_i, x_k]. \end{aligned}$$

Thus by induction, any element of B' can be expressed as a linear combination of elements of B. Hence B is a generating set for $H_1(E')$.

To show that there are no relations among the elements of B we will employ the following map. Let X be the wedge of n circles and let $\widetilde{X}$ denote the universal abelian cover of X. Note that $\pi_1(X) = E$. Thus $\pi_1\left(\widetilde{X}\right) = E'$ and $H_1\left(\widetilde{X}\right) = H_1(E')$. Let v be the vertex of X. Then by the long exact sequence of a pair we have that $i : H_1(E') \hookrightarrow H_1\left(\widetilde{X}, \tilde{v}\right)$, where the i is induced by the natural inclusion. Note that $H_1\left(\widetilde{X}, \tilde{v}\right)$ can be identified with the free $\mathbb{Z}\Lambda$ module with basis $(x_1, \ldots, x_n)$ where Λ is the free abelian group on the basis $(y_1, \ldots y_n)$.

Consider a linear combination $\sum a_{i,j}(k)\ {}^{w_{i,j}(k)}[x_i, x_j]$ of elements of B that is 0 in $H_1(E)$. By looking at the geometry of the inclusion $\widetilde{X} \to \left(\widetilde{X}, \tilde{v}\right)$ we see that $i\left(\sum a_{i,j}(k)\ {}^{w_{i,j}(k)}[x_i, x_j]\right) = \sum a_{i,j}(k) w_{i,j}(k)\left((1 - y_j)x_i + (y_i - 1)x_j\right)$. We define upper and lower bounds on the height of x_i in the y_j direction as follows:

$$U_{i,j} = \max_k \{m_j | w_{i,j}(k) = y_1^{m_1} \cdots y_n^{m_j}, a_{i,j}(k) \neq 0\}$$

$$L_{i,j} = \min_k \{m_j | w_{i,j}(k) = y_1^{m_1} \cdots y_n^{m_j}, a_{i,j}(k) \neq 0\}$$

As the expression $\sum a_{i,j}(k)w_{i,j}(k)\left((1-y_j)x_i + (y_i - 1)x_j\right)$ has finitely many $a_{i,j}(k) \neq 0$ for each pair i, j, if there is $a_{i,j}(k) \neq 0$ for some k, then there must exist words $w_{i,j}(k^*)$ and $w_{i,j}(k_*)$ such that

$$w_{i,j}(k^*) = y_1^{m_1} \cdots y_{j-1}^{m_{j-1}} y_j^{U_{i,j}}$$

$$w_{i,j}(k_*) = y_1^{m_1} \cdots y_{j-1}^{m_{j-1}} y_j^{L_{i,j}}$$

and with $a_{i,j}(k^*)$ and $a_{i,j}(k_*)$ nonzero.

Note that the only parts of the sum $\sum a_{i,j}(k)w_{i,j}(k)\left((1-y_j)x_i + (y_i - 1)x_j\right)$ containing x_1 terms are of the form $a_{1,j}(k)w_{1,j}(k)(1-y_j)x_1$. Suppose that $a_{1,n}(k) \neq 0$ for some k. Consider the $-a_{1,n}(k^*)w_{1,n}(k^*)y_n x_1$ summand of the above expression, $\sum a_{i,j}(k)w_{i,j}(k)\left((1-y_j)x_i + (y_i - 1)x_j\right)$. By assumption, the summation $\sum a_{i,j}(k)w_{i,j}(k)\left((1-y_j)x_i + (y_i - 1)x_j\right) = 0$.

Suppose that $w'_{1,n}(k^*)$ is another word achieving the upper bound $U_{1,n}$. Then $w_{1,n}(k^*) = y_1^{m_1} \cdots y_{j-1}^{m_{n-1}} y_j^{U_{1,n}}$ and $w'_{1,n}(k^*) = y_1^{m'_1} \cdots y_{j-1}^{m'_{n-1}} y_j^{U_{1,n}}$ where $m_i \neq m'_i$ for some $i < n$. Hence the term corresponding to $w'_{1,n}(k^*)$, $a'_{1,n}(k*)w'_{1,n}(k^*)(1-y_n)x_1$ can have no interaction with $-a_{1,n}(k^*)w_{1,n}(k^*)y_n x_1$. The same holds for words $w_{1,n}(k)$ which do not achieve the upper bound $U_{i,j}$.

Note that by definition of our proposed basis $w_{1,j}(k)$ is an element of the free abelian group generated by $y_1, \ldots, y_j$. Hence for $j \neq n$, $w_{1,n}(k^*)y_n x_1 \neq w_{1,j}(k)x_1$ unless $U_{1,n} = -1$.

Similarly, we may consider the $a_{1,n}(k_*)w_{1,n}(k_*)x_1$ summand of the above expression, $\sum a_{i,j}(k)w_{i,j}(k)\left((1-y_j)x_i + (y_i - 1)x_j\right)$. By an analogous argument, for $w_{1,n}(k)x_1 \neq w_{1,n}(k_*)x_1$ and for $j \neq n$, $w_{1,n}(k_*)x_1 \neq w_{1,j}(k)x_1$ unless $L_{1,n} = 0$.

By definition of our upper and lower bounds, $U_{1,n}$ and $L_{1,n}$, $U_{1,n} \geq L_{1,n}$, and hence we cannot have $-1 = U_{1,n} < L_{1,n} = 0$. Thus $a_{1,n}(k) = 0$ for all k.

The key to concluding that $a_{1,n}(k) = 0$ was the fact that for $j \neq n$, $w_{1,n}(k^*)y_n x_1 \neq w_{1,j}(k)x_1$, and $w_{1,n}(k_*)x_1 \neq w_{1,j}(k)x_1$ as the powers of y_n in $w_{1,j}(k)$ are zero. Now that we have concluded $a_{1,n}(k) = 0$ we can make some similar statements about the height of x_1 in the y_{n-1} direction: $w_{1,n-1}(k^*)y_{n-1}x_1 \neq w_{1,j}(k)x_1$, and $w_{1,n-1}(k_*)x_1 \neq w_{1,j}(k)x_1$. Since there are no nonzero $a_{1,n}(k)$, the only terms with nonzero powers of y_{n-1} now come from the $w_{1,n-1}$ terms. We make this concept precise via a descending induction on the index j.

Suppose that the coefficients $a_{1,l}(k) = 0$ for all $l > p$. Suppose that $a_{1,p}(k) \neq 0$ for some k. Consider the $-a_{1,p}(k^*)w_{1,p}(k^*)y_p x_1$ summand of $\sum a_{i,j}(k)w_{i,j}(k)\left((1-y_j)x_i + (y_i - 1)x_j\right)$. By definition of our proposed basis $w_{1,j}(k)$ is an element of the free abelian group generated by $y_1, \ldots, y_j$. Since there are no terms with $j > p$, for $j \neq p$, $w_{1,p}(k^*)y_p x_1 \neq w_{1,j}(k)x_1$ unless $U_{1,p} = -1$. As our sum is zero, the $w_{1,p}(k^*)y_p x_1$ term must be zero. Since the only contribution to this term is the $-a_{1,p}$ coefficient, it follows that $U_{1,p} = -1$.

Similarly, we may consider the $a_{1,p}(k_*)w_{1,p}(k_*)x_1$ summand of the sum, $\sum a_{i,j}(k)w_{i,j}(k)\left((1-y_j)x_i + (y_i - 1)x_j\right)$. As there are no terms with $j > p$, for $j \neq p$,

$w_{1,p}(k_*)x_1 \neq w_{1,j}(k)x_1$ unless $L_{1,p} = 0$. As our sum is zero, the $w_{1,p}(k_*)y_px_1$ term must be zero. Since the only contribution to this term is the $a_{1,p}$ coefficient, it follows that $L_{1,p} = 0$. By definition of our upper and lower bounds, $U_{1,p}$ and $L_{1,p}$, $U_{1,p} \geq L_{1,p}$, and hence we cannot have $-1 = U_{1,p} < L_{1,p} = 0$. Thus $a_{1,p}(k) = 0$ for all k. Hence by our induction, $a_{1,j}(k) = 0$ for all j and all k.

Since $a_{1,j}(k) = 0$, x_2 now plays the role of x_1, and hence we can make the conclusion that $a_{2,j}(k) = 0$. We make this precise by an induction on the index i.

Suppose that $a_{l,j}(k) = 0$ for all j and for all $l < p$. Then the only x_p terms come from the $a_{p,j}(k)(1-y_j)x_p$ summands of $\sum a_{i,j}(k)w_{i,j}(k)\left((1-y_j)x_i + (y_i - 1)x_j\right)$. Thus we can repeat the above induction on the index j to conclude that $a_{p,j}(k) = 0$ for all j and k. Hence $a_{i,j}(k) = 0$ for all i, j and k.

Thus we have shown that for a linear combination $\sum a_{i,j}(k)\ {}^{w_{i,j}(k)}[x_i, x_j]$ of elements of B that is 0 in $H_1(E)$, all $a_{i,j}(k) = 0$. Thus there are no relations in the among the elements in the set B.

As B generates $H_1(E')$ and has no relations, B is a basis for $H_1(E')$.

$\square$

Lemma 4.5. *The module $\frac{E'_k}{E'_{k+1}}$ has no $\mathbb{Z}\left[\frac{E}{E'}\right]$ torsion of the form $(1-x_i)\omega = 0$, where x_i is a generator for E.*

Proof. Consider the homology basis B for E' given by Lemma 4.4. Note that the set of basis elements, B, maps to itself under conjugation by x_1. Given an element $\omega \in \frac{E'_k}{E'_{k+1}}$, by Corollary 4.4, ω can be written as a product $\omega = \prod_{i=1}^m c_i^{\alpha_i}$ where c_i are basic commutators in the elements of $H_1(F')$ and $c_i < c_{i+1}$ for all i.

As in the proof of Lemma 4.4, let X be the wedge of n circles and let $\widetilde{X}$ denote the universal abelian cover of X. Note that $\pi_1(X) = E$. Thus $\pi_1\left(\widetilde{X}\right) = E'$ and $H_1\left(\widetilde{X}\right) = H_1(E')$. Let v be the vertex of X. Then by the long exact sequence of a pair we have that $i : H_1(E') \hookrightarrow H_1\left(\widetilde{X}, \tilde{v}\right)$, where the i is induced by the natural inclusion. The universal abelian cover, $\tilde{X}$, is a 1-complex taking the form of a square grid with a dimension corresponding to each generator. The vertices of this grid can be labeled by a vector $(a_1, a_2, \dots, a_n)$ where a_i denotes the distance of the vertex in the x_i direction. These vertices can be ordered by the dictionary order. That is, if $v = (a_1, \dots, a_n)$ and $v' = (a'_1, \dots, a'_n)$ then $v < v'$ if $a_1 < a'_1$ or $a_i = a'_i$ for all $i < j$ and $a_j < a'_j$.

Any edge in the lattice can then be written as an ordered pair of vertices, $e = (v_1, v_2)$ with $v_1 < v_2$. The edges then inherit a strict ordering by the dictionary order on the weighted vertices. That is, if $e = (v_1, v_2)$ and $e' = (v'_1, v'_2)$ then $e < e'$ if $v_1 < v'_1$ or if $v_1 = v'_1$ and $v_2 < v'_2$.

Any basis element of $H_1(E')$ (in the basis described in Lemma 4.4) can be written as a finite sum of edges, $c = \sum_{i=1}^{l} b_i e_i$ where $e_i \leq e_{i+1}$. Thus the weighting on oriented edges of E' induces an ordering on basis elements of the homology of E' by the dictionary order in the same way. We may represent any element c as a vector $(b_1, b_2, \cdots)$ where b_i is the coefficient of the edge e_i. Note that by construction, such a vector has finitely many nonzero entries. For $c = (b_1, \dots)$ and $c' = (b'_1, \dots)$ then $c < c'$ if $b_1 < b'_1$ or $b_i = b'_i$ for all $i < j$ and $b_j < b'_j$. In this manner we obtain a strict ordering on basis elements of $H_1(E)$. We can use this ordering of our homology basis to construct our basic commutators as in the basis theorem. Conjugating by

x_1 preserves the ordering on homology elements of E' as it preserves the ordering on vertices. Hence since B is invariant under conjugation by x_1, each basic commutator c_i in the product $\omega = \prod_{i=1}^{m} c_i^{\alpha_i}$ where c_i is sent to another basic commutator in the elements of B, c_i'. As the ordering on elements of B is preserved under the conjugation by x_1, the c_i' also satisfy $c_i' < c_{i+1}'$.

Hence by Corollary 4.4, $x_1 \cdot \omega$ can be written uniquely by the basis theorem as $x_1 \cdot \omega = \prod_{i=1}^{m} c_i'^{\alpha_i}$, where $c_i' = {}^{x_1}c_i$. Note that $\omega \neq x_1 \cdot \omega$ as $c_1 \neq c_1'$ and thus the unique expressions of ω and ${}^{x_1}\omega$ have distinct least commutators. Thus $(1 - x_1)\omega \neq 0$.

The result that $(1-x_i)\omega \neq 0$ for any i can be obtained by reordering the generators of E such that x_i plays the role of x_1. $\square$

Note that $(1 - x_1)(1 - x_2)\omega \neq 0$ is equivalent to the statement that the map $\cdot(1 - x_1)(1 - x_2) : \frac{E_k'}{E_{k+1}'} \to \frac{E_k'}{E_{k+1}'}$ is injective. Hence if $\omega \neq \omega'$ then $(1 - x_1)(1 - x_2)\omega \neq (1 - x_1)(1 - x_2)\omega'$. This fact will be employed in future Magnus homomorphism computations.

Chapter 5

Main Results

In this chapter we investigate properties of the higher-order Magnus subgroups. Section 5.1 develops a way of obtaining mapping classes in $M_k(S)$ from those in $J_k(D)$ and shows these mapping classes to be nontrivial. Given that it is known that $J_k(D)$ is nontrivial for all k, this shows that the higher dimensional analog $M_k(S)$ is nontrivial for all k for genus ≥ 3. In Section 5.2 we seek to strengthen this result. We will show that the Magnus homomorphisms are nontrivial on $M_k(S)$ given some conditions on $\tau_k(J_k(D))$. Using these Magnus homomorphism computations we will exhibit a subgroup of $\frac{M_k}{M_{k+1}}$ isomorphic to a lower central series quotient of free groups. Finally, in Section 5.3 we will show that $\frac{M_k(S)}{M_{k+1}(S)}$ is infinite rank for all k.

5.1 Constructing elements of M_k via subsurfaces

Let S be a surface with genus $g \geq 2$ and 1 boundary component. Let D be a sphere with n disks removed, $n \geq 3$. We work to relate Johnson filtration on D to the

Magnus filtration on S by considering separating embeddings of D in S.

Definition 5.1. Let D have boundary components $b_0, \ldots, b_n$. The map $i : D \to S$ is a separating embedding if i is an embedding such that $i(b_1), \ldots, i(b_n)$ are separating curves in S and $i(b_0)$ is either a separating curve in S or is the boundary component of S.

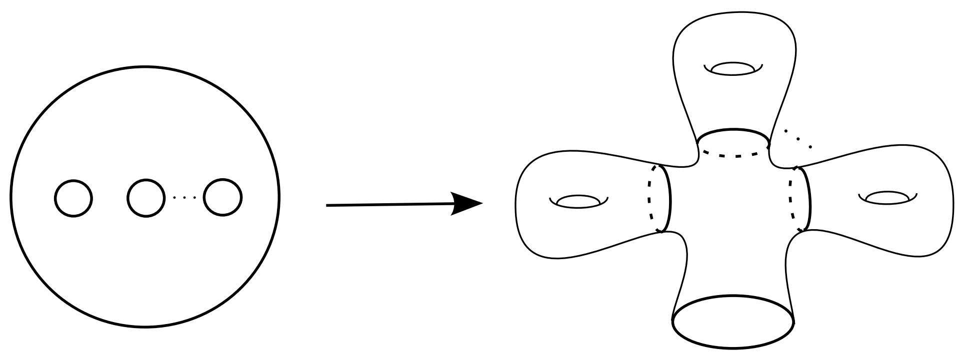

Figure 5.1: To obtain examples of $f' \in M_k(S)$ from $f \in J_k(D)$, we embed the disk, D into S such that each boundary component of D is either separating in S or is the boundary component of S. The above illustrates a possible separating embedding of D_g in S_g.

We first develop a relationship between the Johnson subgroups on a disk and the Magnus subgroups on a larger surface. For this we will employ a specific basis for F that is compatible with the arcs which generate $H_1(D, \partial D)$. Let $*$ be a basepoint for $F = \pi_1(S)$ located on the boundary of S. Let A_i be arcs connecting the i^{th} boundary component to p_0 as in Definition 2.3. Let p_i be the terminal point of A_i. As the boundary components of $i(D)$ are separating in S, $S \setminus D$ is a disjoint union of at most $n + 2$ surfaces, one of which is $i(D)$. Let us denote the other surfaces $\Sigma_0, \ldots, \Sigma_n$, with Σ_0 chosen such that Σ_0 contains the boundary component of S (note that if i maps a boundary component of D to the boundary component of S, Σ_0 is empty). Let Σ_i have genus g_i. Then $\pi_1(\Sigma_i, p_i)$ has a basis consisting of $2g_i$ loops (given the extra boundary component, Σ_0 will have a basis of $2g_0 + 1$ loops, but we will only consider the $2g_0$ loops which form a basis for the capped off surface). By the Seifert Van Kampen theorem, we can combine these bases to form a basis for F as follows: Let C be an arc joining $*$ to p_0. The elements of our basis for S are the homotopy classes of the loops $CA_i\beta\overline{A_i}\,\overline{C}$ (or $C\beta\overline{C}$ for $i = 0$) where β is a generator of $\pi_1(\Sigma_i, p_i)$. This basis is illustrated in Figure 5.2 below. We denote the elements of this basis $\{\alpha_1, \gamma_1, \ldots, \alpha_g, \gamma_g\}$ where the curves $\alpha_{g_0+\cdots+g_{i-1}+1}, \gamma_{g_0+\cdots+g_{i-1}+1}, \ldots, \alpha_{g_0+\cdots+g_i}, \gamma_{g_0+\cdots+g_i}$ are those basis elements produced by the generators of $\pi_1(\Sigma_i, p_i)$.

Lemma 5.1. *Let $i : D \to S$ be a separating embedding. Let $[f] \in Mod(D)$ and let f be a representative homeomorphism of $[f]$. Let $f' : S \to S$ be the homeomorphism defined by*

$$f'(x) = \begin{cases} i(f(y)) & x = i(y) \\ x & x \in S \setminus i(D) \end{cases}$$

then if $[f] \in J_k(D)$, $[f'] \in M_k(S)$.

Proof. Choose an ordering of the boundary components of D, points p_i on these boundary components and arcs A_i as in Definition 2.3 such that the boundary component of S is contained in the component of $S \setminus \mathrm{int} i(D)$ containing the 0^{th} boundary component of D. Let $*$ be a basepoint for $\pi_1(S)$ which lies on ∂S and let c be an arc parametrized on $[0, 1]$ such that $c(0) = *$ and $c(1) = p_0 \in \partial D$. By construction of our basis for $\pi_1(S, *)$ in which each generator can be represented by a loop α which is

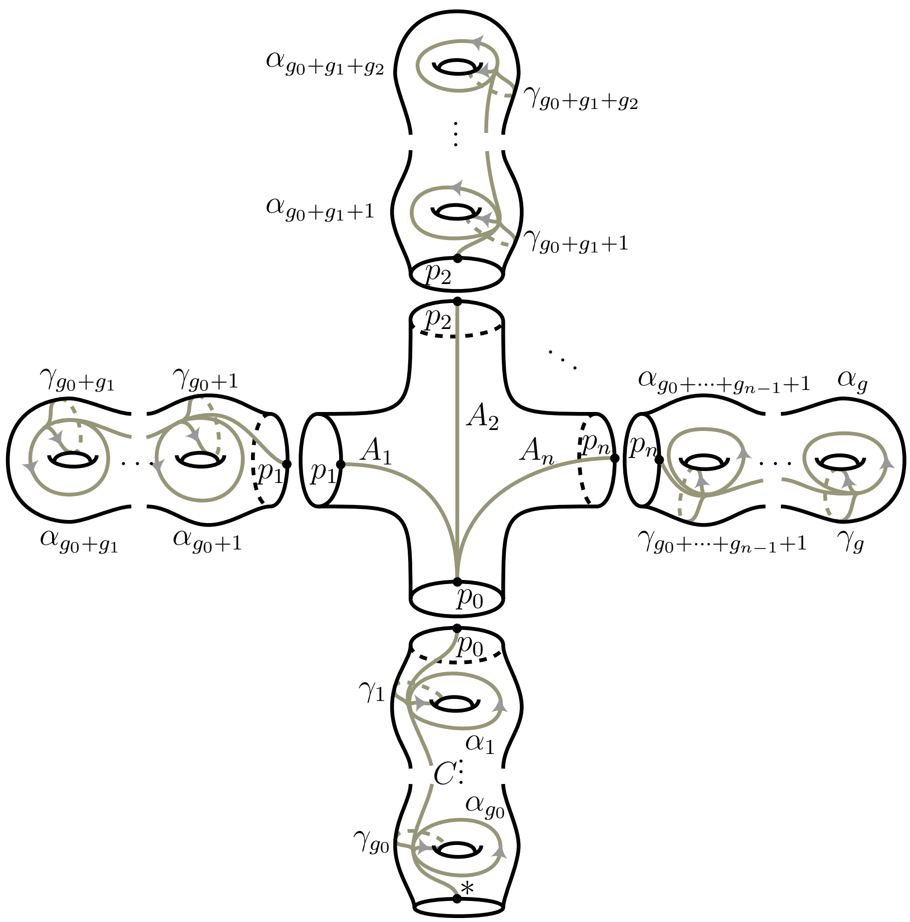

Figure 5.2: Pictured above is the chosen basis $\{\alpha_1, \gamma_1, \ldots, \alpha_g, \gamma_g\}$ of F, obtained by connecting the bases for $\pi_1(\Sigma_i, p_i)$ to the basepoint $*$ via the arcs A_i.

either disjoint from $i(D)$, or is of the form $\alpha = CA_i\beta\overline{A_i}\,\overline{C}$ with β a loop intersecting $i(D)$ only at its initial and terminal points.

For α disjoint from $i(D)$, $f'_*(\alpha) = \alpha$ and thus $f'_*(\alpha)\alpha^{-1} = 1$ is trivially contained in F_k.

For $\alpha = CA_i\beta\overline{A_i}\,\overline{C}$ we can perform the following computation.

$$\begin{aligned}
f'(\alpha)\overline{\alpha} &\simeq f'(CA_i\beta\overline{A_i}\,\overline{C})\overline{(CA_i\beta\overline{A_i}\,\overline{C})} \\
&\simeq f'(C)f'(A_i)f'(\beta)f'(\overline{A_i})f'(\overline{C})CA_i\overline{\beta}\,\overline{A_i}\,\overline{C} \\
&\simeq C\,i(f(A_i))\beta\,i(f(\overline{A_i}))\overline{C}CA_i\overline{\beta}\,\overline{A_i}\,\overline{C} \\
&\simeq \left(C\,i(f(A_i))\overline{A_i}\,\overline{C}\right)\left(CA_i\beta\overline{A_i}\,\overline{C}\right)\left(CA_i i(f(\overline{A_i}))\overline{C}\right)\left(C\overline{\beta}\,\overline{A_i}\,\overline{C}\right) \\
&\simeq i_*\left(f(A_i)\overline{A_i}\right)\left(CA_i\beta\overline{A_i}\,\overline{C}\right)i_*\left(A_i f(\overline{A_i})\right)\left(C\overline{\beta}\,\overline{A_i}\,\overline{C}\right)
\end{aligned}$$

Note that $\left(f(A_i)\overline{A_i}\right)^{-1} = A_i f(\overline{A_i})$. As $f \in J_k(D)$, both $A_i f(\overline{A_i})$ and $f(A_i)\overline{A_i}$ are contained in $\pi_1(D)_k$. Each boundary curve of $i(D)$ is the boundary of a subsurface of S and hence is contained in $[F, F]$. Since $\pi_1(D)$ is generated by the boundary curves of D, it follows that $i_*(\pi_1(D)) \subset F'$, and hence $i_*(\pi_1(D)_k) \subset F'_k$. Hence both $i_*(A_i f(\overline{A_i}))$ and $i_*(f(A_i)\overline{A_i})$ are contained in F'_k. Considering the above expression modulo F'_k we then achieve the following.

$$\begin{aligned}
f'(\alpha)\alpha^{-1} &= CA_i\beta\overline{A_i}\,\overline{C}CA_i\overline{\beta A_i}\,\overline{C} && \text{mod } F'_k \\
&= \alpha\alpha^{-1} = 1 && \text{mod } F'_k
\end{aligned}$$

Therefore $f' \in M_k(S)$.

$\square$

Lemma 5.1 allows us to construct numerous examples of elements of $M_k(S)$ by extending homeomorphisms in J_k of embedded disks.

Proposition 5.2. *Let $i : D \to S$ be a separating embedding. The map $i' : Mod(D) \to Mod(S)$ given by $i'([f]) = [f']$ is an injective homomorphism. This map induces a* *homomorphism*

$$\bar{i} : \frac{J_k(D)}{J_{k+1}(D)} \to \frac{M_k(S)}{M_{k+1}(S)}.$$

Proof. We begin by showing that i' is multiplicative. Consider maps $[f_1], [f_2] \in Mod(D)$ and let f_1, f_2 be corresponding homeomorphisms. Clearly as elements of $Mod(D)$, $[f_1][f_2] = [f_1 f_2]$. The composition $f_1 f_2$ is a representative of the class $[f_1 f_2]$. We then have:

$$\begin{aligned} i'\left([f_1][f_2]\right) &= i'\left([f_1 f_2]\right) \\ &= \left[(f_1 f_2)'\right]. \end{aligned}$$

Note that by definition $(f_1 f_2)'$ is the homeomorphism $S \to S$ which extends $f_1 f_2$ by the identity. We then have that $(f_1 f_2)' = f_1' f_2'$. By definition of multiplication in $Mod(S)$, $[f_1' f_2'] = [f_1'][f_2']$. Thus,

$$\begin{aligned} i'\left([f_1][f_2]\right) &= \left[(f_1 f_2)'\right] \\ &= [f_1' f_2'] \\ &= [f_1'][f_2'] \\ &= i'\left([f_1]\right) i'\left([f_2]\right) \end{aligned}$$

Thus i' is multiplicative.

To show that i' is injective, it then suffices to show that $\ker i' = 1$. This amounts to showing that beginning with a nontrivial mapping class $f \in M_k(D)$, the resulting mapping class $f' \in M_k(S)$ is necessarily nontrivial. As no boundary component of

D is nullhomotopic in S, this follows directly from [5] Theorem 3.18. Hence i' is a monomorphism.

We now address the second part of the proposition: that i' induces a homomorphism $\bar{i} : \frac{J_k(D)}{J_{k+1}(D)} \to \frac{M_k(S)}{M_{k+1}(S)}$. By Lemma 5.1 $i'(J_{k+1}(D)) \subset M_{k+1}(S)$. Thus the map $\bar{i}$ is well defined. It is clearly a homomorphism as i' is a homomorphism. This completes the proof. □

5.2 Magnus homomorphism computations

Having developed a relationship between Johnson subgroups on D and Magnus subgroups on S, we now seek to relate the Johnson homomorphisms on D to the Magnus homomorphisms on S. To do this we must first examine the relationship between the lower central series quotients of $\pi_1(D)$ and F'.

Let G denote the fundamental group of D, the disk with n holes, and let y_i be the generators of G obtained by traveling along arc A_i, circling the corresponding boundary component in a counterclockwise direction, and returning to the basepoint along $\overline{A_i}$ as shown in Figure 5.3.

Lemma 5.3. *Let $i : D \to S$ be a separating embedding. The induced map $i_* : \frac{G_k}{G_{k+1}} \to \frac{F'_k}{F'_{k+1}}$ is injective.*

Proof. To show that the above map is an injection, we will employ Proposition 2.4. Hence we must show that the homomorphism $i_* : G \to F'$ given by $y_i \mapsto [\alpha_i, \gamma_i]$ induces an injection $H_1(G; \mathbb{Q}) \to H_1(F'; \mathbb{Q})$. As G/G' and F'/F'' are torsion free abelian groups, it suffices to show there is an injection $H_1(G; \mathbb{Z}) \to H_1(F'; \mathbb{Z})$. Note

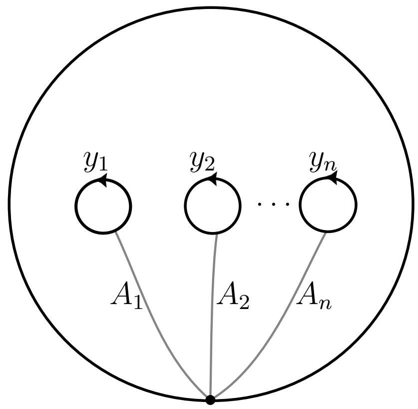

Figure 5.3: Pictured above are the generators y_i of G. A generator y_i is obtained by traveling along arc A_i, circling the corresponding boundary component in a counter-clockwise direction, and returning to the basepoint along $\overline{A_i}$.

that by our previous construction of the basis for F',

$$i_*(y_i) = \left[\alpha_{g_0+\cdots+g_{i-1}+1}, \gamma_{g_0+\cdots+g_{i-1}+1}\right] \cdots \left[\alpha_{g_0+\cdots+g_i}, \gamma_{g_0+\cdots+g_i}\right].$$

Consider an element $\sum n_i y_i$ which is nonzero in G/G'. We compute the image of this element by i_* as follows:

$$\begin{aligned} i_*\left(\sum n_i y_i\right) &= \sum n_i i_*(y_i) \\ &= \sum n_i \left[\alpha_{g_0+\cdots+g_{i-1}+1}, \gamma_{g_0+\cdots+g_{i-1}+1}\right] \cdots \left[\alpha_{g_0+\cdots+g_i}, \gamma_{g_0+\cdots+g_i}\right] \end{aligned}$$

Because $\sum n_i y_i \neq 0$, $n_j \neq 0$ for some j. Consider the map g_j which maps S to the punctured surface as shown in Figure 5.4.

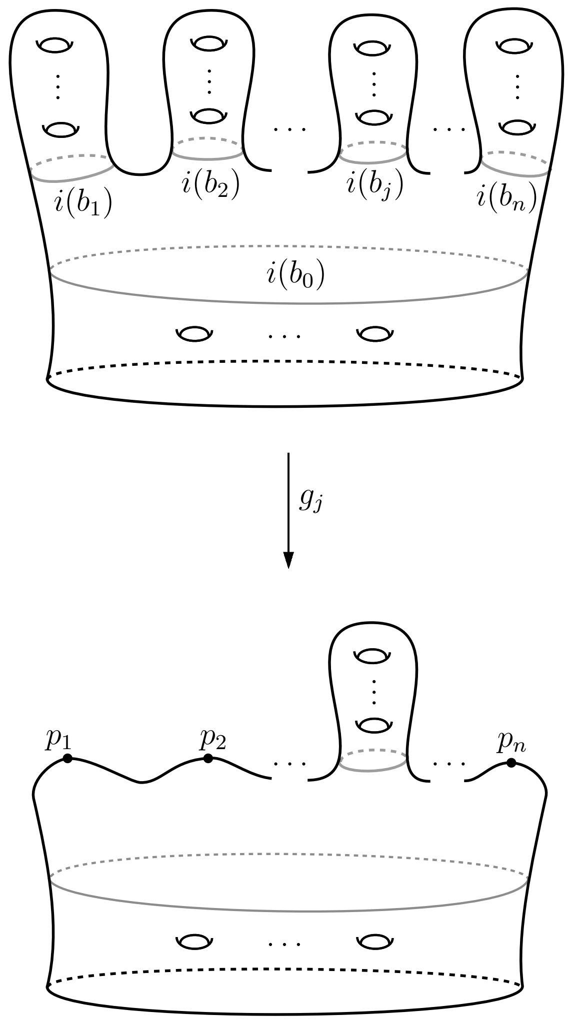

Figure 5.4: Pictured above is the continuous map $g_j : S \to T$. Everything above and including the curve $i(b_i)$ for $i \neq j$ is collapsed to a point p_i.

We find that

$$\begin{aligned} g_{j*}i_*\left(\sum n_i y_i\right) &= \sum n_i i_*(y_i) \\ &= g_{j*}\left(\sum n_i \left[\alpha_{g_0+\cdots+g_{i-1}+1}, \gamma_{g_0+\cdots+g_{i-1}+1}\right]\cdots\left[\alpha_{g_0+\cdots+g_i}, \gamma_{g_0+\cdots+g_i}\right]\right) \\ &= \sum n_i g_{j*}\left(\left[\alpha_{g_0+\cdots+g_{i-1}+1}, \gamma_{g_0+\cdots+g_{i-1}+1}\right]\cdots\left[\alpha_{g_0+\cdots+g_i}, \gamma_{g_0+\cdots+g_i}\right]\right) \\ &= n_j\left[g_{j*}(\alpha_{g_0+\cdots+g_{j-1}+1}), g_{j*}(\gamma_{g_0+\cdots+g_{j-1}+1})\right] \\ &\qquad \cdots\left[g_{j*}(\alpha_{g_0+\cdots+g_j}), g_{j*}(\gamma_{g_0+\cdots+g_j})\right]. \end{aligned}$$

Clearly $n_j\left[g_{j*}(\alpha_{g_0+\cdots+g_{j-1}+1}),g_{j*}(\gamma_{g_0+\cdots+g_{j-1}+1})\right]\cdots\left[g_{j*}(\alpha_{g_0+\cdots+g_j}),g_{j*}(\gamma_{g_0+\cdots+g_j})\right]\neq 0$ in $H_1(T)$. Hence $i_*(\sum n_i y_i)\neq 0$.

$\square$

Let $f' : S \to S$ be constructed by taking a map f in $J_k(D)$ and extending it to the whole surface by the identity, as in Lemma 5.1. We relate the $\tau_k(f)$ and $\tau_k'(f')$ in the following lemma.

Lemma 5.4. *Let S be a surface with genus $g \geq 2$ and 1 boundary component. Let D be a sphere with n disks removed, $n \geq 3$. Let $i : D \to S$ be a separating embedding. Let $f \in J_k(D)$ with $\tau_k(f)(A_i) = w_i \in \pi_1(D)_k/\pi_1(D)_{k+1}$ and let $f' = i'(f)$ be the element of $Mod(S)$ given by Lemma 5.1. Let γ_i and γ_j be elements of F that intersect $i(D)$ along the arcs A_i and A_j respectively. Then $\tau_k'(f')[\gamma_i,\gamma_j] = (1-\gamma_i)(1-\gamma_j)\left(i_*(w_i) - i_*(w_j)\right)$. Furthermore, if $w_i \neq w_j$ as elements of $\pi_1(D)_k/\pi_1(D)_{k+1}$ for some choice of i,j then $\tau_k'(f') \neq 0$.*

Proof. Let $w_i = \tau_k(f)(A_i)$. We wish to show that $\tau_k'(f')[\gamma_i,\gamma_j] \neq 0$. We begin by computing $\tau_k'(f')([\gamma_i,\gamma_j])$. By construction, $\gamma_i = CA_i\beta_i\overline{A_i}\,\overline{C}$ for some loop β_i in $S \setminus D$

by our construction of the basis for F. Then by definition

$$\begin{aligned}
\tau_k'(f')([\gamma_i,\gamma_j]) &= f'([\gamma_i,\gamma_j])[\gamma_j,\gamma_i] \\
&= [f'(\gamma_i), f'(\gamma_j)][\gamma_j,\gamma_i] \\
&= [f'(CA_i\beta_i\overline{A_i}\,\overline{C}), f'(CA_j\beta_j\overline{A_jC})][\gamma_j,\gamma_i] \\
&= [C\,if(A_i)\beta_i if(\overline{A_i})\overline{C}, C\,if(A_j)\beta_j if(\overline{A_j})\overline{C}][\gamma_j,\gamma_i] \\
&= [C\,if(A_i)\beta_i\overline{if(A_i)}\overline{C}, C\,if(A_j)\beta_j\overline{if(A_j)}\overline{C}][\gamma_j,\gamma_i] \\
&= \Big[C\,i\left(f(A_i)\overline{A_i}\right)A_i\beta_i\overline{A_i}i\left(A_i\overline{f(A_i)}\right)\overline{C}, \\
&\qquad C\,i\left(f(A_j)\overline{A_j}\right)A_j\beta_j\overline{A_j}i\left(A_j\overline{f(A_j)}\right)\overline{C}\Big][\gamma_j,\gamma_i] \\
&= \Big[\left(C\,i\left(f(A_i)\overline{A_i}\right)\overline{C}\right)\left(CA_i\beta_i\overline{A_i}\,\overline{C}\right)\left(C\,i\left(A_i\overline{f(A_i)}\right)\overline{C}\right), \\
&\qquad \left(C\,i\left(f(A_j)\overline{A_j}\right)\overline{C}\right)\left(CA_j\beta_j\overline{A_jC}\right)\left(Ci\left(A_j\overline{f(A_j)}\right)\overline{C}\right)\Big][\gamma_j,\gamma_i].
\end{aligned}$$

Note that $i_* : \pi_1(D,p_0) \to \pi_1(S,i(p_0))$. Allowing a change of basepoint from $\pi_1(S,i(p_0))$ to $\pi_1(S,*) = F$, we may further reduce this expression as follows:

$$\begin{aligned}
\tau_k'(f')([\gamma_i,\gamma_j]) &= [i_*(w_i)\gamma_i i_*(w_i^{-1}), i_*(w_j)\gamma_j i_*(w_j^{-1})][\gamma_j,\gamma_j] \\
&= [[i_*(w_i),\gamma_i]\gamma_i, [i_*(w_j),\gamma_j]\gamma_j][\gamma_j,\gamma_i]
\end{aligned}$$

Using the commutator identities $[ga,b] = {}^{g}[a,b][g,b]$ and $[a,hb] = [a,h]\,{}^{h}[a,b]$ it is possible to reduce this expression to the following:

$$\begin{aligned}
\tau_k'(f')([\gamma_i,\gamma_j]) = {}^{[i_*(w_i),\gamma_i]}[\gamma_i,[i_*(w_j),\gamma_j]]\ {}^{[i_*(w_i),\gamma_i][i_*(w_j),\gamma_j]}[\gamma_i,\gamma_j][[i_*(w_i),\gamma_i],[i_*(w_j),\gamma_j]] \\
{}^{[i_*(w_j),\gamma_j]}[[i_*(w_i),\gamma_i],\gamma_j][\gamma_j,\gamma_i].
\end{aligned}$$

As $i_*(G) \subset F'$ and $w_i, w_j \in G_k$, $i_*(w_i), i_*(w_j) \in F'_k$. Thus the commutators $[i_*(w_i), \gamma_i], [i_*(w_j), \gamma_j]$ are elements of F'_k and hence the conjugation in our expression is trivial modulo F'_{k+1}. In addition $[[i_*(w_i), \gamma_i], [i_*(w_j), \gamma_j]] \in F'_{k+1}$. Thus reducing mod F'_{k+1} we obtain:

$$\begin{aligned}\tau'_k(f')([\gamma_i, \gamma_j]) &= [\gamma_i, [i_*(w_j), \gamma_j]][\gamma_i, \gamma_j][[i_*(w_i), \gamma_i], \gamma_j][\gamma_j, \gamma_i] \\ &= [\gamma_i, [i_*(w_j), \gamma_j]][[i_*(w_i), \gamma_i], \gamma_j]\end{aligned}$$

Equivalently, viewing the $\tau'_k(f')([\gamma_i, \gamma_j])$ as an element of the $\mathbb{Z}[F/F']$ module we can represent it as follows:

$$\tau'_k(f')[\gamma_i, \gamma_j] = (1 - \gamma_i)(1 - \gamma_j)\,(i_*(w_i) - i_*(w_j))$$

where $(1 - \gamma_i), (1 - \gamma_j) \in \mathbb{Z}[F/F]$ and $i_*(w_i), i_*(w_j) \in F'_k/F'_{k+1}$. This proves the first statement of the lemma.

To prove that $w_i w_j^{-1} \neq 0$ shows $\tau'_k(f') \neq 0$, we find it advantageous to express the above computation as follows:

$$\tau'_k(f')[\gamma_i, \gamma_j] = (1 - \gamma_i)(1 - \gamma_j)\left(i_*(w_i w_j^{-1})\right).$$

By Lemma 4.5, $\tau'_k(f')[\gamma_i, \gamma_j]$ is nonzero provided that $i_*(w_i w_j^{-1})$ is nontrivial. This follows directly from Lemma 5.3. $\square$

Let D_n be the disk with n holes and let $G(n) = \pi_1(D_n)$. Let $E(n)$ denote the free group generated by $x_1, x_2, \ldots, x_n$. Let $P(n)$ denote the pure braid group on n strands. Consider the inclusion $\iota : E(n-1) \to P(n)$ obtained by mapping the generator x_i of

$E(n-1) = \langle x_1, \ldots, x_{n-1} \rangle$ by $x_i \mapsto A_{i,n}$ where $A_{i,n}$ is the generator of the pure braid group which clasps strands i and n [2] as shown in Figure 5.5.

Forgetting to fix the boundary components in the interior of the disk, any mapping class in $Mod(D)$ is isotopic to the identity. The trace of this isotopy permutes the boundary components on the interior of the disk to generate a pure braid. This correspondence is an isomorphism between $P(n)$ and $Mod(D_n)$. We denote this natural map $\psi : P(n) \to Mod(D_n)$. In particular it is important to note that the pure braid generator $A_{i,n}$ yields a mapping class $f_{i,n}$ on D_n given by a single dehn twist (twisting right) around the i^{th} and n^{th} punctures as shown in Figure 5.5. Note

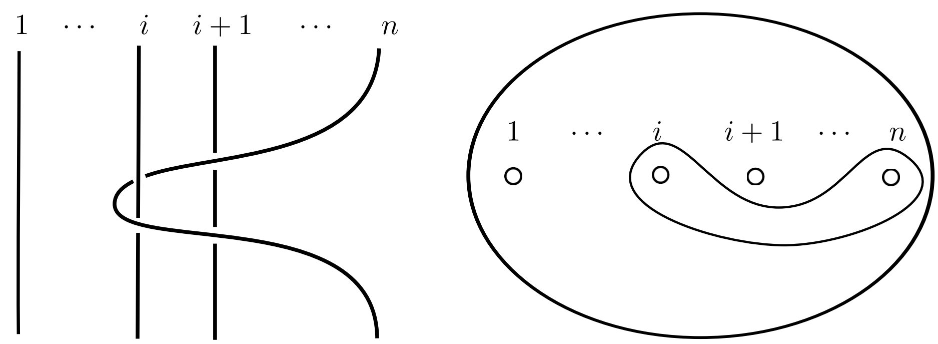

Figure 5.5: Left: The generator $A_{i,n}$ of the pure braid group. Right: The Dehn twist $f_{i,n}$ corresponding to $A_{i,n}$

that as function composition is written right to left, the map ψ acts by reversing the order of pure braid generators: $\psi(A^{\epsilon_1}_{p_1,n} \cdots A^{\epsilon_m}_{p_m,n}) = f^{\epsilon_m}_{p_m,n} \cdots f^{\epsilon_1}_{p_1,n}$.

For a mapping class $f \in Mod(D_n)$, let $\phi_i(f)$ be given by $\phi_i(f) = f(A_i)\overline{A_i}$.

Lemma 5.5. *The map $\theta : E(n-1) \to G(n-1)$ given by the composition of maps*

$$E(n-1) \overset{\iota}{\hookrightarrow} P(n) \overset{\psi}{\hookrightarrow} Mod(D_n) \overset{\phi_n}{\to} G(n) \overset{\pi}{\to} G(n-1)$$

is the isomorphism induced by mapping $x_i \mapsto y_i$.

The map $\mu : E(n-1) \to G(n-1)$ *given by the composition of maps*

$$E(n-1) \overset{\iota}{\hookrightarrow} P(n) \overset{\psi}{\hookrightarrow} Mod(D_n) \overset{\phi_1}{\to} G(n) \to G(n-1)$$

is the homomorphism given by $v \mapsto y_1^{\eta}$ *where* η *is the sum of the* x_1 *exponents in* v.

Proof. To show this it suffices to trace $v \in E(n-1)$ through the above maps. By the above definitions it is clear that for $v = x_{p_1}^{\epsilon_1} \cdots x_{p_m}^{\epsilon_m}$ that $(\psi \circ \iota)(v) = f_{p_m,n}^{\epsilon_m} \cdots f_{p_1,n}^{\epsilon_1}$. Let $(\psi \circ \iota)(v) = f$.

To compute $\phi_n(f)$ and $\phi_1(f)$ we examine the image of the arcs A_1 and A_n, and the generators of $G(n)$ under a map $f_{i,n}$. By direct computation we find that:

$$f_{i,n}(A_1) \simeq \begin{cases} A_1 & \text{if } 1 < i \\ y_n y_1 A_1 & \text{if } i = 1 \end{cases}$$

$$f_{i,n}(A_n) \simeq y_n y_i A_n$$

$$f_{i,n}(y_n) \simeq y_n y_i y_n y_i^{-1} y_n^{-1} \simeq y_n [y_i, y_n]$$

$$f_{i,n}(y_i) \simeq y_n y_i y_n^{-1} \simeq y_i [y_i^{-1}, y_n]$$

$$f_{i,n}(y_j) \simeq \begin{cases} [y_i, y_n]^{-1} y_j [y_i, y_n] & \text{if } i < j, j \neq g \\ y_j & \text{if } i > j \end{cases}$$

Similarly, we can compute the image of the arcs A_1 and A_n, and the generators of $G(n)$ under the map $f_{i,n}^{-1}$, the left handed Dehn twist about the same simple closed curve.

$$f_{i,n}^{-1}(A_1) \simeq \begin{cases} A_1 & \text{if } 1 < i \\ y_1^{-1} y_n^{-1} A_1 & \text{if } i = 1 \end{cases}$$

$$f_{i,n}^{-1}(A_n) \simeq y_i^{-1} y_n^{-1} A_n$$

$$f_{i,n}^{-1}(y_n) \simeq y_i y_n^{-1} y_n y_n y_i \simeq [y_i^{-1}, y_n] y_n$$

$$f_{i,n}^{-1}(y_i) \simeq y_i^{-1} y_n^{-1} y_i y_n y_i \simeq [y_i^{-1}, y_n^{-1}] y_i$$

$$f_{i,n}^{-1}(y_j) \simeq \begin{cases} [y_i^{-1}, y_n^{-1}] y_j [y_i^{-1}, y_n^{-1}]^{-1} & \text{if } i < j, j \neq g \\ y_j & \text{if } i > j \end{cases}$$

Let N be the normal subgroup of $G(n)$ normally generated by y_n. We can rewrite the above computations as follows. We use $f_{i,n}$ and $f_{i,n}^{-1}$ to denote both the mapping classes and their induced map on $G(n)$ for convenience of notation.

$$f_{i,n}(A_1)\overline{A_1} \simeq \begin{cases} 1 & \text{if } 1 < i \\ y_1 & \text{if } i = 1 \end{cases}$$

$$f_{i,n}(A_n)\overline{A_n} \simeq y_i$$

$$f_{i,n}(y_n) \simeq 1$$

$$f_{i,n}(y_j) \simeq y_j \qquad \text{if } j \neq g$$

For the inverse map $f_{i,n}^{-1}$ we compute:

$$f_{i,n}^{-1}(A_1)\overline{A_1} \simeq \begin{cases} 1 & \text{if } 1 < i \\ \\ y_1^{-1} & \text{if } i = 1 \end{cases}$$

$$f_{i,n}^{-1}(A_n)\overline{A_n} \simeq y_i^{-1}$$

$$f_{i,n}^{-1}(y_n) \simeq 1$$

$$f_{i,n}^{-1}(y_j) \simeq y_j \qquad \text{if } j \neq n$$

From the above it is clear that $f_{i,n}$ and $f_{i,n}^{-1}$ act by the identity on the first $n-1$ generators of G, and sends y_n to an element of the subgroup N. Note that the map $G \mapsto G/N$ is the homomorphism of $\pi_1(D_n)$ induced by the map ℓ which caps the n^{th} boundary component of D_n as in Figure 5.6. As the mapping classes $f_{i,n}$ and $f_{i,n}^{-1}$ fix

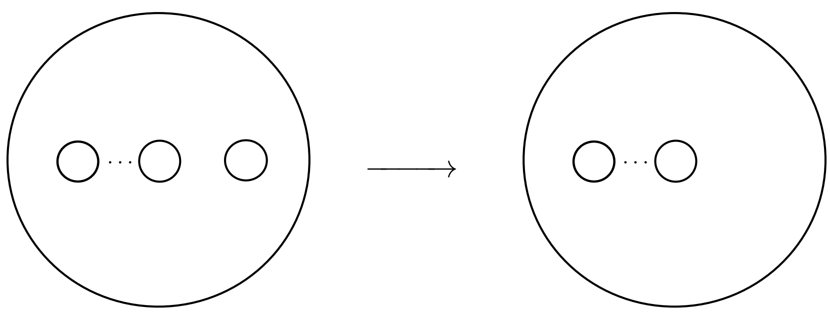

Figure 5.6: Above is an illustration of the map $\ell : D_n \to D_{n-1}$ obtained by capping off the n^{th} boundary component. From this one can see $\pi_1(D_{n-1}) = \langle y_1, \ldots y_{n-1} \rangle \cong G/N$.

the n^{th} boundary component and ℓ is an inclusion map, the map ℓ commutes with $f_{i,n}$ and $f_{i,n}^{-1}$. Hence $N = f_{i,n}(N) = f_{i,n}^{-1}(N)$. Thus, given a word in $v \in G$, $f_{i,n}$ and $f_{i,n}^{-1}$ each map v to a word of the same class in G/N.

We show by induction that for $f = f_{p_m,n}^{\epsilon_m} \cdots f_{p_1,n}^{\epsilon_1}$ the following computations hold mod N:

$$f(A_1)\overline{A_1} = y_1^{\eta}$$

$$f(A_n)\overline{A_n} = y_{p_1}^{\epsilon_1} \cdots y_{p_m}^{\epsilon_m}$$

where $\eta = \sum_{p_i=1} \epsilon_i$.

For simplicity of notation we rewrite these equalities as

$$f(A_1)\overline{A_1} = y_1^{\pm 1} \cdots y_1^{\pm 1}$$

$$f(A_n)\overline{A_n} = y_{l_1}^{\pm 1} \cdots y_{l_k}^{\pm 1}$$

where indexes are allowed to repeat. The initial case of the induction was done by previous computations. Suppose that the above computations hold. Then it follows that

$$f(A_1) \simeq a_0 y_1^{\pm 1} \cdots y_1^{\pm 1} A_1$$

$$f(A_n) \simeq a_0' y_{l_1}^{\pm 1} \cdots y_{l_k}^{\pm 1} A_n.$$

where $a_0, a_0' \in N$. Consider $f_{p_{m+1},n}^{\pm 1} f_{p_m,n}^{\epsilon_m} \cdots f_{p_1,n}^{\epsilon_1} = f_{p_{m+1},n}^{\pm 1} f$. As $f_{p_{m+1},n}^{\pm 1}$ acts by the identity on G/N, we have that

$$f_{p_{m+1},n}(y_i) = a_i^+ y_i$$

$$f^{-1}_{p_{m+1},n}(y_i) = a_i^- y_i$$

for some $a_i^+, a_i^- \in N$, for all i. Hence we compute

$$f_{p_{m+1},n} f(A_1) \simeq \begin{cases} f_{p_{m+1},n}(a_0)\left(a_1^+ y_1\right)^{\pm 1} \cdots \left(a_1^+ y_1\right)^{\pm 1} A_1 & \text{if } p_{m+1} \neq 1 \\ f_{p_{m+1},n}(a_0)\left(a_1^+ y_1\right)^{\pm 1} \cdots \left(a_1^+ y_1\right)^{\pm 1} y_n y_1 A_1 & \text{if } p_{m+1} = 1 \end{cases}$$

$$f^{-1}_{p_{m+1},n} f(A_1) \simeq \begin{cases} f^{-1}_{p_{m+1},n}(a_0)\left(a_1^- y_1\right)^{\pm 1} \cdots \left(a_1^- y_1\right)^{\pm 1} A_1 & \text{if } p_{m+1} \neq 1 \\ f^{-1}_{p_{m+1},n}(a_0)\left(a_1^- y_1\right)^{\pm 1} \cdots \left(a_1^- y_1\right)^{\pm 1} y_1^{-1} y_n^{-1} A_1 & \text{if } p_{m+1} = 1 \end{cases}$$

$$f_{p_{m+1},n} f(A_n) \simeq f_{p_{m+1},n}(a_0')\left(a_{l_1}^+ y_{l_1}\right)^{\pm 1} \cdots \left(a_{l_k}^+ y_{l_k}\right)^{\pm 1} y_n y_{p_{m+1}} A_n$$

$$f^{-1}_{p_{m+1},n} f(A_n) \simeq f_{p_{m+1},n}(a_0')\left(a_{l_1}^- y_{l_1}\right)^{\pm 1} \cdots \left(a_{l_k}^- y_{l_k}\right)^{\pm 1} y_{p_{m+1}}^{-1} y_n^{-1} A_n.$$

Therefore, as $f^{\pm 1}_{p_{m+1},n}(a_0), f^{\pm 1}_{p_{m+1},n}(a_0'), a_i^+, a_i^-, y_n, y_n^{-1} \in N$, we can do the following computation mod N.

$$f^{\pm 1}_{p_{m+1},n} f^{\epsilon_m}_{p_m,n} \cdots f^{\epsilon_1}_{p_1,n}(A_1)\overline{A_1} = \begin{cases} y_1^{\eta} y_1^{\pm 1} & \text{if } p_{m+1} = 1 \\ y_1^{\eta} & \text{if } p_{m+1} \neq 1 \end{cases}$$

$$f^{\pm 1}_{p_{m+1},n} f^{\epsilon_m}_{p_m,n} \cdots f^{\epsilon_1}_{p_1,n}(A_n)\overline{A_n} = y_{p_1}^{\epsilon_1} \cdots y_{p_m}^{\epsilon_m} y_{p_{m+1}}^{\pm 1}$$

This completes the induction.

Thus $\theta(x_{p_1}^{\epsilon_1} \cdots x_{p_m}^{\epsilon_m}) = y_{p_1}^{\epsilon_1} \cdots y_{p_m}^{\epsilon_m}$ and $\mu(w) = y_1^{\eta}$, as desired.

$\square$

Note that for words $v \in [E(n-1), E(n-1)]$ the maps $f_{1,n}$ occur in pairs with opposite exponents. Hence for $v \in [E(n-1), E(n-1)]$, $\mu(v) = 1$.

5.3 Structure of the Magnus subgroup quotients

In Lemma 5.5 we considered compositions of maps which defined a correspondence between elements of the free group $E(n-1)$ and elements of $Mod(D_n)$. Lemma 5.4 allows us to relate the Johnson homomorphism $J_k(D_n)$ to the Magnus homomorphism $M_k(S_g)$. We now combine these tools to construct families of mapping classes in $M_k(S_g)$ which have a desirable algebraic structure in the image of the Magnus homomorphism.

Let $i : D_g \to S_g$ be the separable embedding illustrated in Figure 5.1. Consider the following composition of maps:

$$E(g-1) \overset{\iota}{\hookrightarrow} P(g) \overset{\psi}{\hookrightarrow} Mod(D_g) \xrightarrow{i'} Mod(S_g)$$

where $i' : Mod(D_g) \to Mod(S_g)$ is the map described in Lemma 5.1. Let $\rho = i' \circ \psi \circ \iota$. The following theorem illustrates that ρ retains the structure of the free group.

Theorem 5.6. *Let S be an orientable surface with genus $g \geq 3$. Then the map $\rho : E(g-1) \to Mod(S)$ induces a monomorphism on the quotients $\overline{\rho} : E(g-1)_k/E(g-1)_{k+1} \to M_k(S)/M_{k_1}(S)$ for all k.*

Proof. Let D_g be a disk with g punctures. To prove the theorem it suffices to show that mapping classes contained in the subgroup $\rho(E(g-1))$ satisfy the conditions of Lemma 5.4 and produce distinct images through the Magnus homomorphism. For this we employ several results about the pure braid group, $P(g)$. Consider the following split exact sequence

$$1 \to E(g-1) \to P(g) \to P(g-1) \to 1$$

where the map $E(g-1) \to P(g)$ is as in Lemma 5.5 and $P(g) \to P(g-1)$ is given by forgetting the g^{th} strand. This exact sequence induces an isomorphism as given in [6]:

$$\frac{E(g-1)_k}{E(g-1)_{k+1}} \oplus \frac{P(g-1)_k}{P(g-1)_{k+1}} \cong \frac{P(g)_k}{P(g)_{k+1}}$$

In particular, the map ι induces an injective map $\bar{\iota}$ on the lower central series quotients:

$$\bar{\iota} : \frac{E(g-1)_k}{E(g-1)_{k+1}} \hookrightarrow \frac{P(g)_k}{P(g)_{k+1}}$$

.

By direct analysis of the induced automorphisms on $G(g)$ [2] Corollary 1.8.3, it is clear that $\psi\,(P(g)) \subset J_2(D_g)$. Given this, Lemma 2.2 shows that $\psi\,(P(g))_k \subset J_k(D_g)$. Hence, we have a well defined map $\overline{\psi} : \frac{P(g)_k}{P(g)k+1} \to \frac{J_k(D_g)}{J_{k+1}(D_g)}$. By [7], Theorem 1.1 $\psi\,(P(g)_{k+1}) = \psi\,(P(g)) \cap J_{k+1}(D_g)$. Hence the map $\overline{\psi}$ is injective.

$$\overline{\psi} : \frac{P(g)_k}{P(g)k+1} \hookrightarrow \frac{J_k(D_g)}{J_{k+1}(D_g)}$$

By Proposition 5.2, the map i' induces a monomorphism $\bar{i} : \frac{J_k(D_g)}{J_{k+1}(D_g)} \to \frac{J_k(D_g)}{J_{k+1}(D_g)}$. By Lemma 5.4, given $v \in \frac{E(g-1)_k}{E(g-1)_k}$ we have that

$$\tau_k'(i'\psi\iota(v))[\gamma_1, \gamma_g] = (1-\gamma_1)(1-\gamma_g)i_*(w_1)i_*(w_g)^{-1}$$

written as an element of $\frac{F'_k}{F'_{k+1}}$ as a $\mathbb{Z}\left[\frac{F}{F'}\right]$ module where $w_i = \tau_k(f)(A_i)$.

Note that we have traced $w \in E(g-1)_k/E(g-1)_{k+1}$ through the following composition of maps:

$$\frac{E(g-1)_k}{E(g-1)_k} \overset{\bar{\iota}}{\hookrightarrow} \frac{P(g)_k}{P(g)_{k+1}} \overset{\overline{\psi}}{\hookrightarrow} \frac{J_k(D_g)}{J_{k+1}(D_g)} \overset{\bar{i}}{\hookrightarrow} \frac{M_k(S_g)}{M_{k+1}(S_g)} \xrightarrow{\tau'_k(-)[\gamma_1,\gamma_g]} \frac{F'_k}{F'_{k+1}}$$

.

By definition, the maps $\bar{\iota}$ and $\overline{\psi}$ are homomorphisms. Proposition 5.2 shows that $\bar{i}$ is a homomorphism. The map $\tau'_k(-)[\gamma_1, \gamma_g] : M_k(S) \to F'_k/F'_{k+1}$ is a homomorphism as τ'_k is a homomorphism. As all maps in this composition are homomorphisms, the composition map is also a homomorphism.

As this composition is a homomorphism, in order to complete the proof it suffices to show that the image of v through this composition is not the identity. As shown in Lemma 4.5, this module has no torsion of the form $(1-\gamma)x = 0$ where γ is a generator of F. Hence for $i_*(w_1)i_*(w_g)^{-1} = i_*(w_1 w_g^{-1}) \neq 1$ as an element of F'_k/F'_{k+1}, we have that $\tau'_k(i'\psi\iota(w))[\gamma_1, \gamma_g] = (1-\gamma_1)(1-\gamma_g)i_*(w_1 w_g^{-1}) \neq 0$. By Lemma 5.3 the map i_* is injective, thus it suffices to show that $w_1 w_g^{-1} \neq 1$ as an element of $\frac{G(g)_k}{G(g)_{k+1}}$. By Lemma 5.5,

$$\begin{aligned} \pi(w_1 w_g^{-1}) &= \pi(w_1)\pi(w_g)^{-1} \\ &= \mu(v)\theta(v)^{-1} \\ &= v^{-1} && \text{when written in the generators } y_i \text{ of } G(g-1). \end{aligned}$$

Since π is a homomorphism we can conclude that $w_1 w_g^{-1} \neq 1$ and hence

$\tau'_k(i'\psi\iota(w))[\gamma_1, \gamma_g] = (1-\gamma_1)(1-\gamma_g)i_*(w_1 w_g^{-1}) \neq 0$. This shows that $\ker(\overline{\rho}) = 0$ and hence $\overline{\rho}$ is injective.

$\square$

Theorem 5.7. *Let S be an orientable surface with genus $g \geq 3$. Then the successive quotients of the Magnus filtration $\frac{M_k(S)}{M_{k+1}(S)}$ surject onto an infinite rank torsion free abelian subgroup of $\frac{F'_k}{F'_{k+1}}$ via the map*

$$\frac{M_k(S)}{M_{k+1}(S)} \xrightarrow{\tau'_k(-)[c_6,c_2]} \frac{F'_k}{F'_{k+1}}$$

where c_2 and c_6 are the generators of F illustrated in Figure 5.8.

Proof. Let γ and δ_n be the simple closed curves on S shown in Figure 5.7. Let $i_n : D \to S$ be the embedding which sends the 3 holed disk D to a neighborhood $\gamma \cup \delta_n$. This set of embeddings of the disk onto S was used by Church and Farb in [3], Theorem 3.2 to produce an infinite family of mapping classes in $Mag(S)$. We employ the same embeddings to produce an infinite family of mapping classes in $M_k(S)$.

Let the free group $E(2)$ be generated by $\{x_1, x_2\}$. Consider the commutator $c^k = [\cdots[[x_2, x_1], x_1], \cdots, x_1] \in E(2)_k$ (commutator with x_1 $k-1$ times). By [7] Theorem 1.1, this commutator yields a nontrivial element of $\frac{J_k(D)}{J_{k+1}(D)}$ through the composition:

$$E(2) \overset{\iota}{\hookrightarrow} P(3) \overset{\psi}{\hookrightarrow} Mod(D_3)$$

Let f_k be the mapping class in $\frac{J_k(D)}{J_{k+1}(D)}$ which arises from the commutator c^k: $f_k = \iota\psi(c^k)$. Let $i'_n(f_k)$ be the mapping class of S resulting from extending f_k by the identity on S using the embedding i_n.

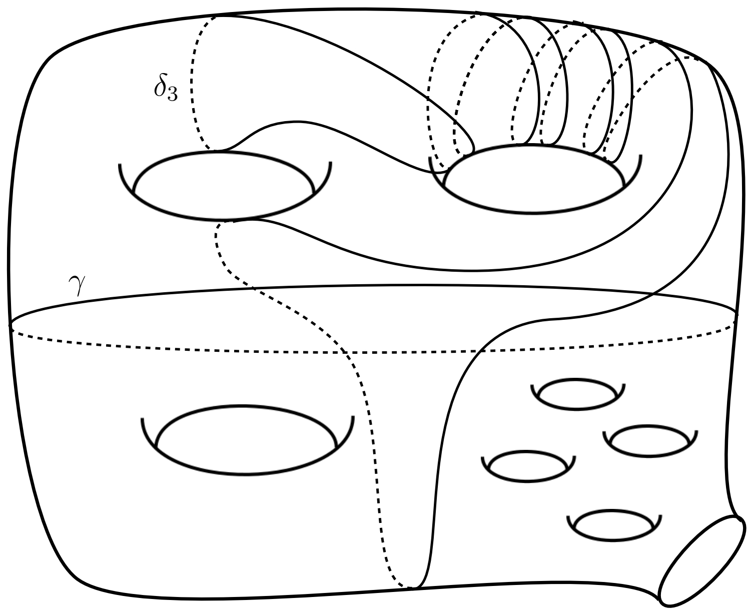

Figure 5.7: Pictured above are two simple closed curves γ and δ_3. The curve δ_n wraps n times around the upper right handle. We consider disks with 3 holes embedded by maps i_n which send D to a neighborhood of $\gamma \cup \delta_n$.

Each embedding $i_n : D \to S$ yields an infinite family of elements $\tau'_k(i'_n(f_k))[c_6, c_2]$. We will show that for each k the set $\{\tau'_k(i'_n(f_k))[c_6, c_2] | n \in \mathbb{N}\}$ is independent in $\frac{F'_k}{F'_{k+1}}$ using the basis theorems developed in Section 4.1.

We begin by choosing a basis for F for our computations. Our chosen basis is illustrated in Figure 5.8. Note that the generators c_2 and c_6 intersect each embedding $i_n(D)$ along the arcs A_1 and A_3 respectively. Hence, we may compute $\tau'_k(i'_n(f_k))[c_6, c_2]$ as in Lemma 5.4.

By Lemma 5.4,

$$\tau'_k(i'_n(f_k))[c_6, c_2] = (1 - c_6)(1 - c_2)i_{n*}(w^k_3(w^k_1)^{-1})$$

where $w^k_i = f_k(A_i)\overline{A_i}$. To show the set $\{\tau'_k(i'_n(f_k))[c_6, c_2] | n \in \mathbb{N}\}$ is independent we

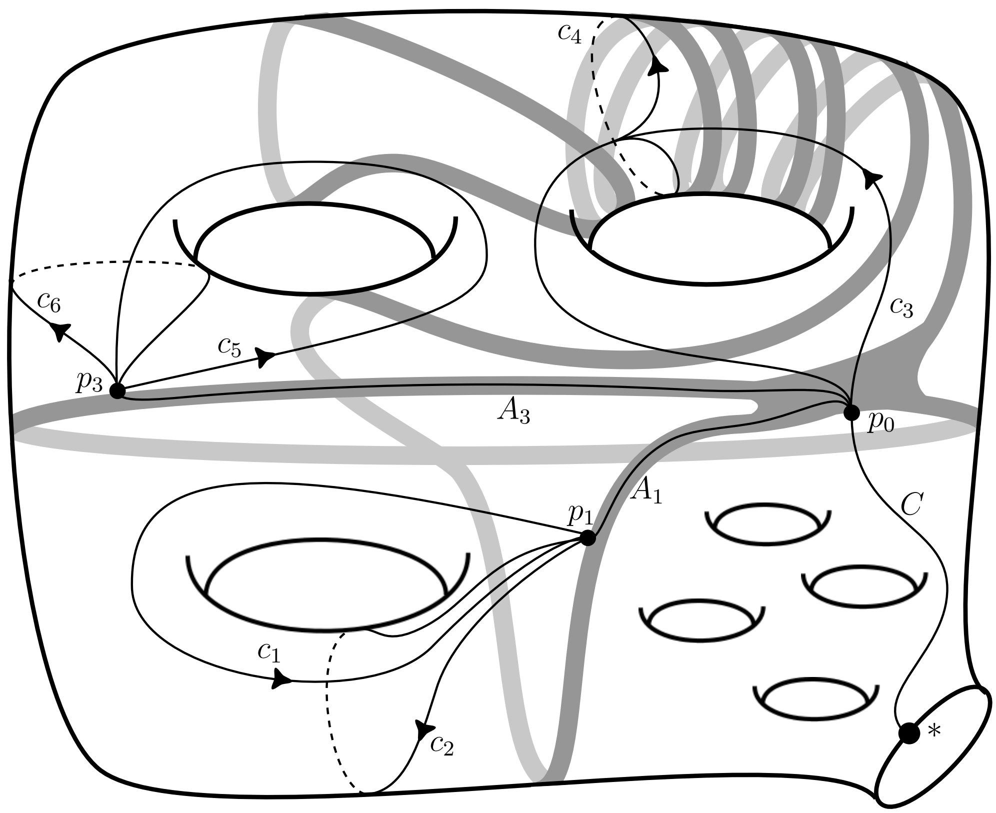

Figure 5.8: The subsurface $i_3(D) \subset S$ is shown in grey. The figure illustrates the basis $\{c_1, c_2, c_3, c_4, c_5, c_6, \ldots, c_{2g}\}$ chosen for computation of the Magnus homomorphisms. Note that c_2 and c_6 intersect $i_n(D)$ along the arcs A_i as in Lemma 5.4.

must compute the elements $i_{n*}(w_1)$ and $i_{n*}(w_3)$. By Lemma 4.5 the set $\{\tau_k'(i_n'(f_k))[c_6, c_2]|n \in \mathbb{N}\}$ is independent if $\{i_{n*}(w_3^k(w_1^k)^{-1})|n \in \mathbb{N}\}$ is an independent set in $\frac{F_k'}{F_{k+1}'}$.

We impose the following ordering the elements of our basis for F': $c_1 < c_2 < c_3 < c_4 < c_5 < c_6$. Then by Lemma 4.4 the set $B = \{ \, ^{w_{i,j}}[c_i, c_j]|w_{i,j} \in H_1\left(E(c_1, \ldots, c_j)\right)\}$ is a basis for F'. By Corollary 4.4, for each $n \in N$, $i_{n*}(w_3^k(w_1^k)^{-1})$ can be expressed as a product of basic commutators of weight k in the generators of B. To show that the set $\{i_{n*}(w_3^k(w_1^k)^{-1})\}_{n \in \mathbb{N}}$ is independent we work towards expressing the elements as basic commutators in our basis B.

We denote the generators of G which loop around the 3 interior boundary components of D counterclockwise by y_1, y_2, y_3 as in Lemma 5.5. As shown in [3], Theorem 3.2, for the embedding $i_n : D \to S$ the generators of G map to the following elements of F written in terms of the basis chosen basis for $\pi_1(S)$:

$$\begin{aligned} i_{n*}(y_1) &= [c_2, c_1] \\ i_{n*}(y_2) &= {}^{[c_5,c_6][c_3,c_4]c_4}[c_3c_4^{-1}c_3^{-1}c_6, c_5c_6^n] \\ i_{n*}(y_3) &= [c_4, c_5c_6^n c_3]. \end{aligned}$$

Again, we have allowed a change of basepoint from $\pi_1(S, p_0)$ to $\pi_1(S, *)$ in this computation.

Recall that $\pi : G(3) \to G(2)$ is the map obtained by taking the quotient by the normal subgroup generated by y_3. The retract $\pi : G(3) \to G(2)$ induces a retract of the lower central series quotients $\overline{\pi} : \frac{G(3)_k}{G(3)_{k+1}} \to \frac{G(2)_k}{G(2)_{k+1}}$. Let $j : G(2) \to G(3)$ be the natural inclusion map. Thus $\overline{\pi}\overline{j} : \frac{G(2)_k}{G(2)_{k+1}} \to \frac{G(2)_k}{G(2)_{k+1}}$ is the identity map. By Lemma 5.5 we have $\pi(w_1^k) = 1$ and $\pi(w_3^k) = [\cdots[[y_2, y_1], y_1]\cdots, y_1]$. Thus $\pi(w_3(w_1)^{-1}) = \pi(w_3)$. It then follows that $w_3^k(w_1^k)^{-1} = j\pi(w_3^k)\eta^k$ for some $\eta^k \in \ker \pi$.

We now compute the elements $i_{n*}j\pi(w_3^k(w_1^k)^{-1})$ using the above expressions for

$i_{n*}(y_1)$ and $i_{n*}(y_2)$.

$$\begin{aligned}
i_{n*}j\pi(w_3^k(w_1^k)^{-1}) &= i_{n*}j\pi(w_3^k) \\
&= (i_{n*}([\cdots[[y_2,y_1],y_1],\cdots,y_1])) \\
&= ([\cdots[[i_{n*}(y_2),i_{n*}(y_1)],i_{n*}(y_1)],\cdots,i_{n*}(y_1)]) \\
&= \left(\left[\cdots\left[\left[\,{}^{[c_5,c_6][c_3,c_4]c_4}[c_3c_4^{-1}c_3^{-1}c_6,c_5c_6^n],[c_2,c_1]\right],[c_2,c_1]\right],\cdots,[c_2,c_1]\right]\right)
\end{aligned}$$

We will compute the elements $i_{n*}j\pi(w_3^k(w_1^k)^{-1})$ explicitly in terms of this basis B. To do this we must reduce the expressions for $i_{n*}(y_2)$ to products of basis elements of F'/F''. Employing the commutator identity $[ga,b] = {}^g[a,b][g,b]$ we can re-write the element $[c_3c_4^{-1}c_3^{-1}c_6,c_5c_6^n]$ as follows:

$$\begin{aligned}
i_{n*}(y_2) &= [c_3c_4^{-1}c_3^{-1}c_6,c_5c_6^n] \\
&= {}^{c_3}[c_4^{-1}c_3^{-1}c_6,c_5c_6^n][c_3,c_5c_6^n] \\
&= {}^{c_3c_4^{-1}}[c_3^{-1}c_6,c_5c_6^n]\ {}^{c_3}[c_4^{-1},c_5c_6^n][c_3,c_5c_6^n] \\
&= {}^{c_3c_4^{-1}c_3^{-1}}[c_6,c_5c_6^n]\ {}^{c_3c_4^{-1}}[c_3^{-1},c_5c_6^n]\ {}^{c_3}[c_4^{-1},c_5c_6^n][c_3,c_5c_6^n]
\end{aligned}$$

Using the commutator identity $[a,vb] = [a,v]\ {}^v[a,b]$, for any element c we have:

$$\begin{aligned}
[c,c_5c_6^n] &= [c,c_5]\ {}^{c_5}[c,c_6^n] \\
&= [c,c_5]\ {}^{c_5}[c,c_6]\ {}^{c_5c_6}[c,c_6^{n-1}] \\
&= [c,c_5]\ {}^{c_5}[c,c_6]\ {}^{c_5c_6}[c,c_6]\cdots\ {}^{c_5c_6^{n-1}}[c,c_6]
\end{aligned}$$

Using this, our original expression becomes:

$$\begin{aligned}
i_{n*}(y_2) = {}& {}^{c_3c_4^{-1}c_3^{-1}}[c_6, c_5] \; {}^{c_3c_4^{-1}c_3^{-1}c_5}[c_6, c_6] \; {}^{c_3c_4^{-1}c_3^{-1}c_5c_6}[c_6, c_6] \cdots {}^{c_3c_4^{-1}c_3^{-1}c_5c_6^{n-1}}[c_6, c_6] \\
& {}^{c_3c_4^{-1}}[c_3^{-1}, c_5] \; {}^{c_3c_4^{-1}c_5}[c_3^{-1}, c_6] \; {}^{c_3c_4^{-1}c_5c_6}[c_3^{-1}, c_6] \cdots {}^{c_3c_4^{-1}c_5c_6^{n-1}}[c_3^{-1}, c_6] \\
& {}^{c_3}[c_4^{-1}, c_5] \; {}^{c_3c_5}[c_4^{-1}, c_6] \; {}^{c_3c_5c_6}[c_4^{-1}, c_6] \cdots {}^{c_3c_5c_6^{n-1}}[c_4^{-1}, c_6] \\
& [c_3, c_5] \; {}^{c_5}[c_3, c_6] \; {}^{c_5c_6}[c_3, c_6] \cdots {}^{c_5c_6^{n-1}}[c_3, c_6].
\end{aligned}$$

As $[c_6, c_6] = 1$ this expression automatically reduces. Using the identity $[a^{-1}, b] = {}^{a^{-1}}[b, a]$ we simplify further to the following expression

$$\begin{aligned}
i_{n*}(y_2) = {}& {}^{c_3c_4^{-1}c_3^{-1}}[c_6, c_5] \\
& {}^{c_3c_4^{-1}c_3^{-1}}[c_5, c_3] \; {}^{c_3c_4^{-1}c_5c_3^{-1}}[c_6, c_3] \; {}^{c_3c_4^{-1}c_5c_6c_3^{-1}}[c_6, c_3] \cdots {}^{c_3c_4^{-1}c_5c_6^{n-1}c_3^{-1}}[c_6, c_3] \\
& {}^{c_3c_4^{-1}}[c_5, c_4] \; {}^{c_3c_5c_4^{-1}}[c_6, c_4] \; {}^{c_3c_5c_6c_4^{-1}}[c_6, c_4] \cdots {}^{c_3c_5c_6^{n-1}c_4^{-1}}[c_6, c_4] \\
& [c_3, c_5] \; {}^{c_5}[c_3, c_6] \; {}^{c_5c_6}[c_3, c_6] \cdots {}^{c_5c_6^{n-1}}[c_3, c_6].
\end{aligned}$$

Noting that $[a, b] = [b, a]^{-1}$ we can now write $i_{n*}(y_2)$ (additively) as follows:

$$\begin{aligned}
i_{n*}(y_2) = {}& - {}^{c_3c_4^{-1}c_3^{-1}}[c_5, c_6] - {}^{c_3c_4^{-1}c_3^{-1}}[c_3, c_5] - \sum_{i=0}^{n-1} {}^{c_3c_4^{-1}c_5c_6^{i}c_3^{-1}}[c_3, c_6] \\
& - {}^{c_3c_4^{-1}}[c_4, c_5] - \sum_{i=0}^{n-1} {}^{c_3c_5c_6^{i}c_4^{-1}}[c_4, c_6] + [c_3, c_5] + \sum_{i=0}^{n-1} {}^{c_5c_6^{i}}[c_3, c_6].
\end{aligned}$$

By Proposition 4.3, for a fixed n we may write $i_{n*}(j\pi w_3^k)$ in terms of basic commutators in the generators of B as follows.

$$\begin{aligned}
i_{n*}\left(j\pi w_3^k\right) = & - \left[\cdots \left[{}^{c_3c_4^{-1}c_3^{-1}}[c_5, c_6], [c_2c_1]\right], \cdots [c_2, c_1]\right] \\
& - \left[\cdots \left[{}^{c_3c_4^{-1}c_3^{-1}}[c_3, c_5], [c_2c_1]\right], \cdots [c_2, c_1]\right] \\
& - \sum_{i=0}^{n-1} \left[\cdots \left[{}^{c_3c_4^{-1}c_5c_6^{i}c_3^{-1}}[c_3, c_6], [c_2c_1]\right], \cdots [c_2, c_1]\right] \\
& - \left[\cdots \left[{}^{c_3c_4^{-1}}[c_4, c_5], [c_2c_1]\right], \cdots [c_2, c_1]\right] \\
& - \sum_{i=0}^{n-1} \left[\cdots \left[{}^{c_3c_5c_6^{i}c_4^{-1}}[c_4, c_6], [c_2c_1]\right], \cdots [c_2, c_1]\right] \\
& + \left[\cdots [[c_3, c_5], [c_2c_1]], \cdots [c_2, c_1]\right] \\
& + \sum_{i=0}^{n-1} \left[\cdots \left[{}^{c_5c_6^{i}}[c_3, c_6], [c_2c_1]\right], \cdots [c_2, c_1]\right].
\end{aligned}$$

By Proposition 4.3, $\ker \overline{\pi}$ is generated by weight k basic commutators in the generators y_1, y_2, y_3 with y_3 in at least one entry. For convenience of notation, let us denote the elements of B by a_i. Let $A \subset B$ be the set of all elements a_i such that a_i appears with a nonzero coefficient in the expression for $i_{n*}(y_3)$ for some $n \in N$ when written in terms of the basis B. Let Y be the subgroup of $\frac{F'_k}{F'_{k+1}}$ generated by basic commutators with an entry from the set A. Note that by construction, $i_{n*}(\ker \overline{\pi}) \subset Y$ for each n. Hence if the elements $i_{n*}(w_3^k(w_1^k)^{-1})$ are independent in $\frac{F'_k/F'_{k+1}}{Y}$ they are also independent in F'_k/F'_{k+1}. Also notice that by construction, the group $\frac{F'_k/F'_{k+1}}{Y}$ is a free abelian group generated by basic commutators in elements of $B \setminus A$. To consider whether the elements $i_{n*}(w_3^k(w_1^k)^{-1})$ are independent in $\frac{F'_k/F'_{k+1}}{Y}$ we must first determine the set A.

We begin by simplifying the expression for $i_{n*}(y_3)$ using the commutator identity

$[a, vb] = [a, v]\ {}^{v}[a, b]$ as follows:

$$\begin{aligned} i_{n*}(y_3) &= [c_4, c_5c_6^nc_3] \\ &= [c_4, c_5]\ {}^{c_5}[c_4, c_6^nc_3] \\ &= [c_4, c_5]\ {}^{c_5}[c_4, c_6^n]\ {}^{c_5c_6^n}[c_4, c_3] \\ &= [c_4, c_5]\ {}^{c_5}[c_4, c_6]\ {}^{c_5c_6}[c_4, c_6^{n-1}]\ {}^{c_5c_6^n}[c_4, c_3] \\ &= [c_4, c_5]\ {}^{c_5}[c_4, c_6]\ {}^{c_5c_6}[c_4, c_6] \cdots\ {}^{c_5c_6^{n-1}}[c_4, c_6]\ {}^{c_5c_6^n}[c_4, c_3]. \end{aligned}$$

Note that the term ${}^{c_5c_6^n}[c_4, c_3]$ is not an element of our chosen basis for $H_1(F', \mathbb{Z})$ as $c_6 > c_4$. In order to express $i_{n*}(y_3)$ in terms of our basis for $H_1(F', \mathbb{Z})$ we rewrite this term as follows:

$$\begin{aligned} {}^{c_5c_6^n}[c_4, c_3] =& [c_3, c_4][c_4, c_5][c_3, c_5]\ {}^{c_4}[c_1, c_5]\ {}^{c_3}[c_4, c_5] \prod_{i=0}^{n-1}\ {}^{c_5c_6^i}[c_3, c_6] \prod_{i=0}^{n-1}\ {}^{c_5c_6^i}[c_3, c_6] \\ & \prod_{i=0}^{n-1}\ {}^{c_5c_3c_6^i}[c_4, c_6] \prod_{i=0}^{n-1}\ {}^{c_5c_4c_6^i}[c_3, c_6]. \end{aligned}$$

Collecting the basis elements of B that occur in the above expressions for $i_{n*}(y_3), n \in N$ we find A to be the following set:

$$A = \left\{ \begin{array}{c} [c_4, c_5],\ {}^{c_5}[c_4, c_6],\ {}^{c_5c_6^i}[c_4, c_6], [c_3, c_4], [c_4, c_5], [c_3, c_5], \\ {}^{c_4}[c_1, c_5],\ {}^{c_3}[c_4, c_5],\ {}^{c_5c_6^i}[c_3, c_6],\ {}^{c_5c_3c_6^i}[c_4, c_6],\ {}^{c_5c_4c_6^i}[c_3, c_6] \end{array} \middle|\ i \in \mathbb{N} \right\}$$

Note that by construction, when viewed as elements of $\frac{F'/F''}{Y}$,

$$\begin{aligned} i_{n*}(w_3^k(w_1^k)^{-1}) &= i_{n*}(j\pi(w_3^k)\eta^k) \\ &= i_{n*}(j\pi(w_3^k)). \end{aligned}$$

Thus the elements $i_{n*}(w_3^k(w_1^k)^{-1})$ are independent in F'/F'' if the elements $i_{n*}(j\pi(w_3^k))$ are independent in $\frac{F'/F''}{Y}$.

Modulo Y, $i_{n*}(j\pi(w_3^k))$ can be written as follows:

$$\begin{aligned} i_{n*}\left(j\pi w_3^k\right) = &- \left[\cdots\left[\,{}^{c_3c_4^{-1}c_3^{-1}}[c_5,c_6],[c_2c_1]\right],\cdots[c_2,c_1]\right] \\ &- \left[\cdots\left[\,{}^{c_3c_4^{-1}c_3^{-1}}[c_3,c_5],[c_2c_1]\right],\cdots[c_2,c_1]\right] \\ &- \sum_{i=0}^{n-1}\left[\cdots\left[\,{}^{c_3c_4^{-1}c_5c_6^{i}c_3^{-1}}[c_3,c_6],[c_2c_1]\right],\cdots[c_2,c_1]\right] \\ &- \left[\cdots\left[\,{}^{c_3c_4^{-1}}[c_4,c_5],[c_2c_1]\right],\cdots[c_2,c_1]\right] \\ &- \sum_{i=0}^{n-1}\left[\cdots\left[\,{}^{c_3c_5c_6^{i}c_4^{-1}}[c_4,c_6],[c_2c_1]\right],\cdots[c_2,c_1]\right]. \end{aligned}$$

Consider a finite sum of these elements. In $\frac{F'_k/F'_{k+1}}{Y}$, this sum can be written as $\sum_{m=1}^{M}\alpha_m i_{n_m*}(j\pi(w_3^k))$ where $\alpha_m \neq 0$ and $n_m < n_{m+1}$ for all m. The M^{th} term of this product is the only term containing a multiple of the basis element $\left[\cdots\left[\,{}^{c_3c_5c_6^{M-1}c_4^{-1}}[c_6,c_4],[c_2c_1]\right],\cdots[c_2,c_1]\right]$. Hence the sum cannot be trivial, and thus the elements $i_{n*}(j\pi(w_3^k))$ must be independent in $\frac{F'_k/F'_{k+1}}{Y}$. Therefore the elements $i_{n*}(w_3^k(w_1^k)^{-1})$ are independent in $\frac{F'_k}{F'_{k+1}}$.

As the set $\{i_{n*}(w_3^k(w_1^k)^{-1})^{\alpha_n}|n \in \mathbb{N}\}$ is an independent set in F'_k/F'_{k+1}, the set $\{(1-c_6)(1-c_2)i_{n*}(w_3^k(w_1^k)^{-1})\}$ is also an independent set. As $\tau'_k(i'_n(f_k))[c_6,c_2] = (1-c_6)(1-c_2)i_{n*}(w_3^k(w_1^k)^{-1})$, this shows that $\frac{M_k(S)}{M_k(S)}$ surjects onto an infinite rank torsion free abelian subgroup of F'_k/F'_{k+1} via the map $f \mapsto \tau'_k(f)[c_6,c_2]$.

□

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