Knotting and linking in 4-dimensions.

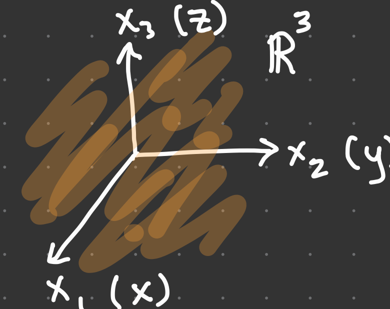
Prof. Shelly Harvey

(Low dimensional) Topology

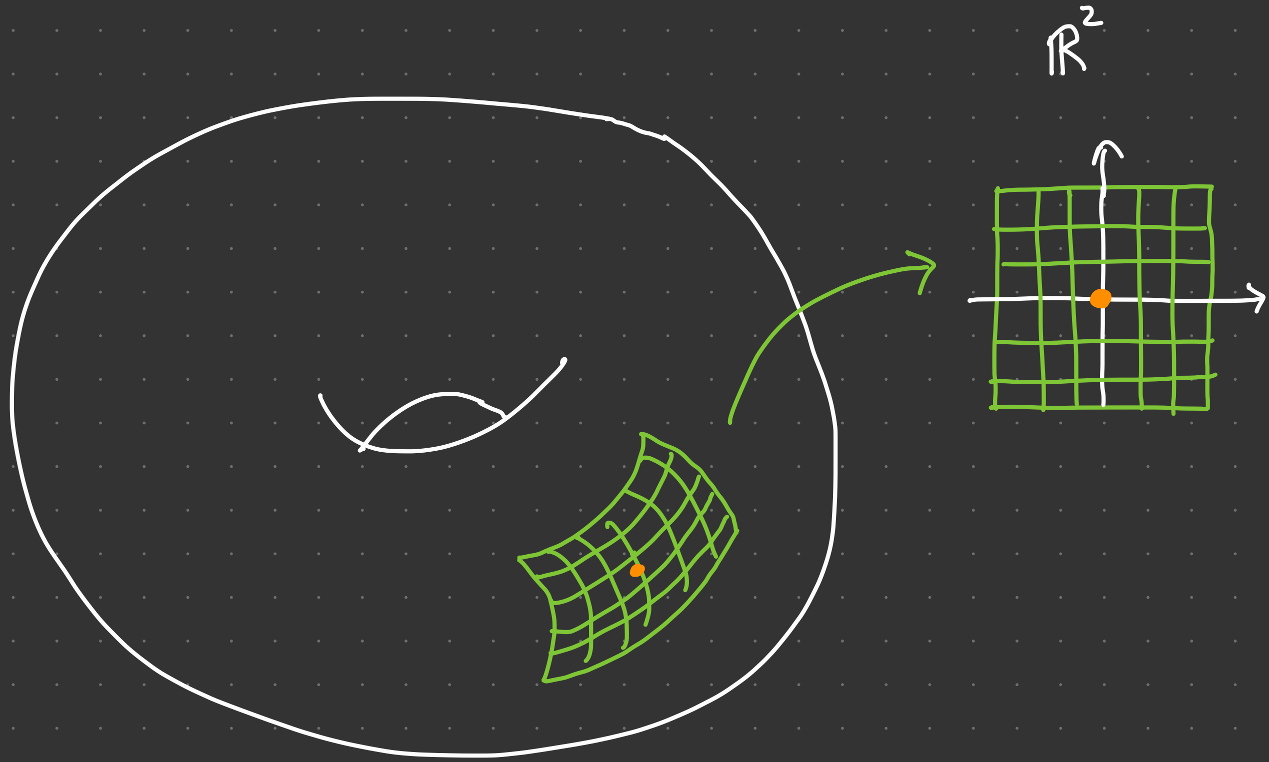
Study spaces that locally look like

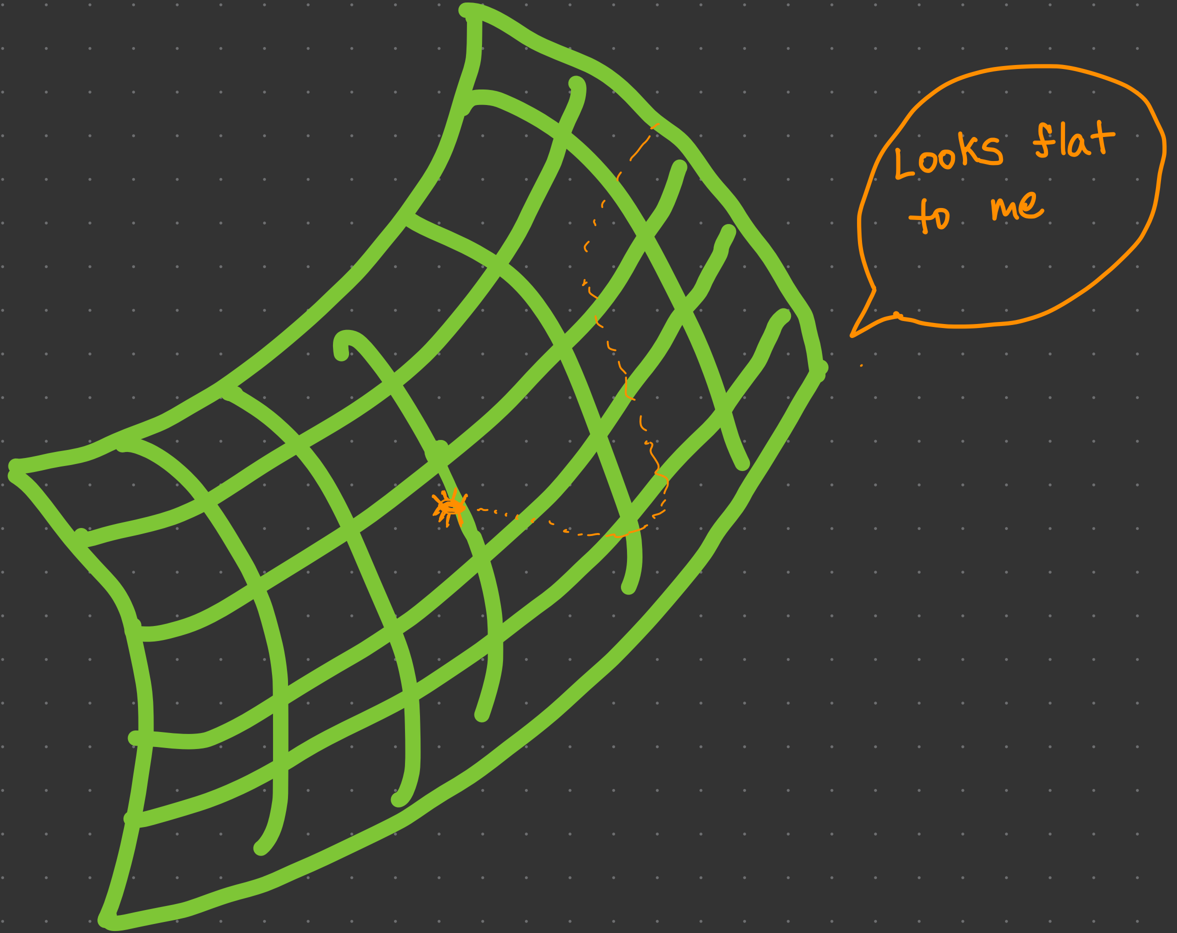
$$\mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R}\}$$

called n-dimensional manifolds



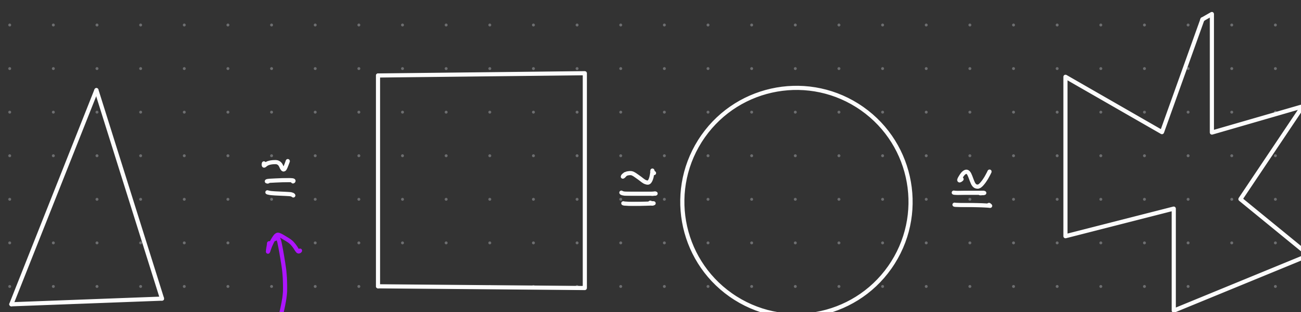
Ex $n=2$ (surfaces)


$\mathbb{R}^2$


Looks flat to me

In topology, we are allowed to bend, twist, stretch spaces and we think of them as the sam

i.e. distances and angles are not preserved



same in topology (not in geometry)

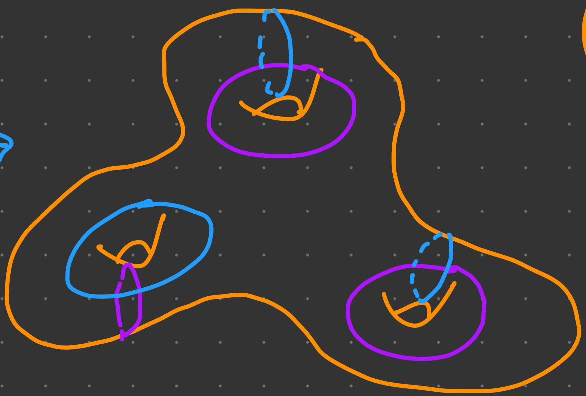
Pop quiz:

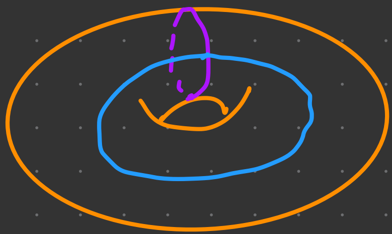
Which spaces are the same?

How to distinguish them?

consider "knots" inside of them.

6 "essential" curves





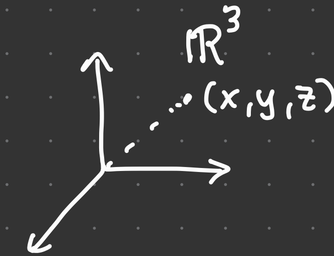
2 "essential" curves

What about higher dimensions?

$n=3$ 3-dimensions

<u>locally</u> looks like $\mathbb{R}^3 = \{(x,y,z) \mid x,y,z \in \mathbb{R}\}$

can't "draw" in the usual sense



What types of objects live in 3D?

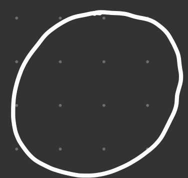
knotted circles

knotted surfaces

Def: A **knot** in $\mathbb{R}^3$ is a "smooth" simple closed curve in $\mathbb{R}^3$.

(or locally flat)

Exs:



unknot

left-handed trefoil

figure eight

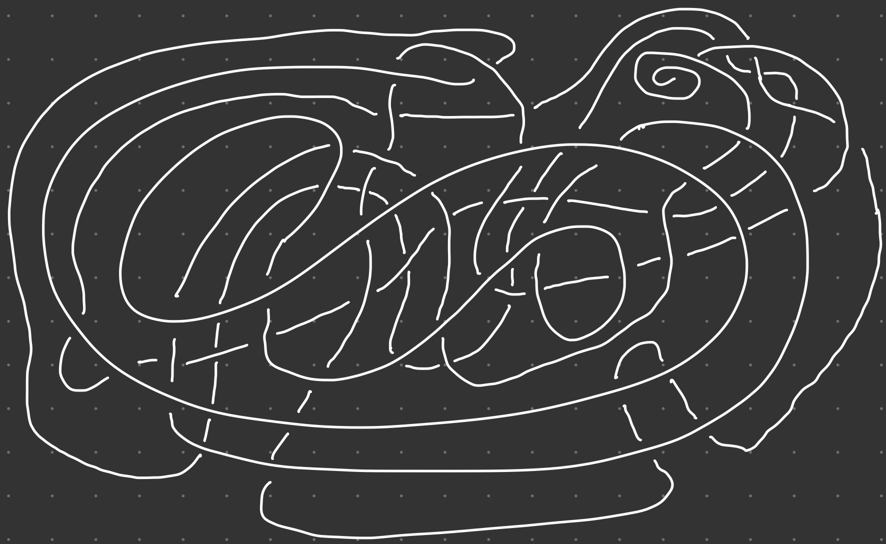
. . .

Not allowed:

We say two knots are the same (isotopic) if we can continuously deform one to the other through knots (i.e. never cut or pass the knot through itself).

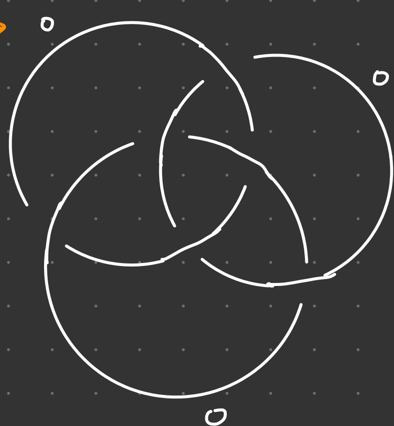
Quiz: Do you recognize this knot?

The diagram / picture of a knot can be complicated!

 =  !

<u>Remark</u>: Any (compact, without boundary) 3-dimensional manifold can be "drawn" (not unique) as a weighted link (knot with multiple components).

can be rational numbers



$= S^1 \times S^1 \times S^1$

↑ 3-dimensional torus

How can we distinguish knots?

A: Knot invariants.

input knot

knot invariant computer

integer
matrix
polynomial
group
ring
field

Important: if two knots are the same, they should have the same output!

Two types of invariants

1. Defined using diagram $\leftarrow$ related to algebra.

Let $cr(K) = \#$ of crossing in a diagram.

Q. Is this a knot invariant?

$c(K) = \min\{\#$ of crossing in a diagram for $K\}$.

* $c(K) = 0 \iff K =$ unknot.

* Only knots with $c(K) = 3$ is trefoil

* $c(K) = 4 \Rightarrow K =$ figure-eight

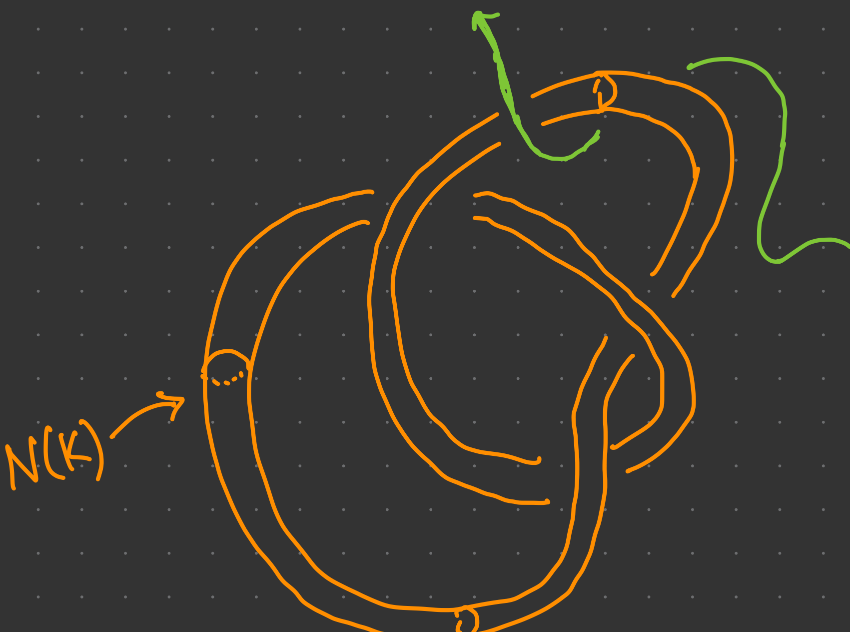
* $c(K) = 5$ more than one!

(need to use other invariants to distinguish)

2. Define using the 3-dimensional manifold

$$\mathbb{R}^3 \setminus N(K)$$

thickened neighborhood of K



can fly around in space but must avoid the knot.

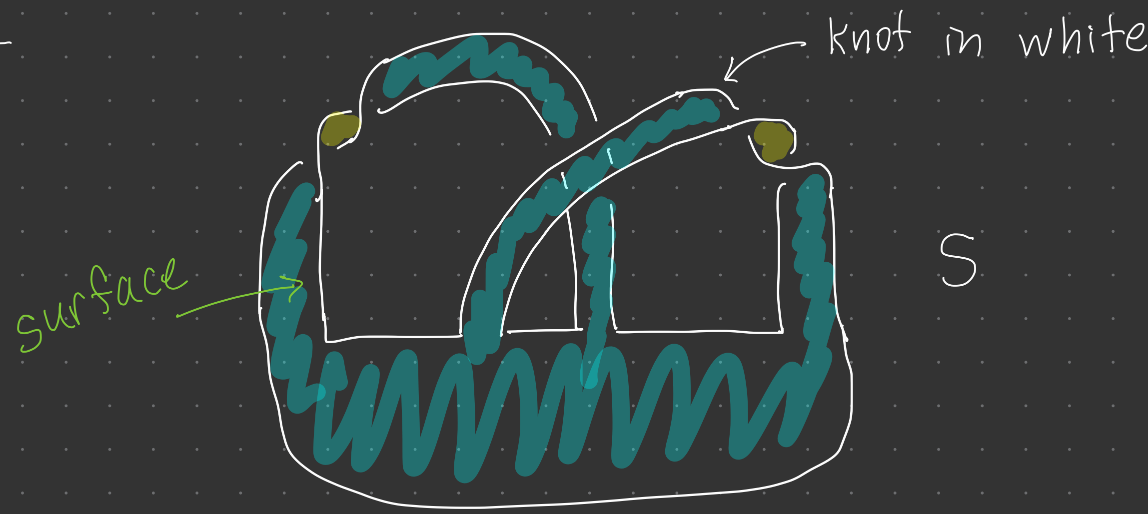
Ex: Fundamental group of K.

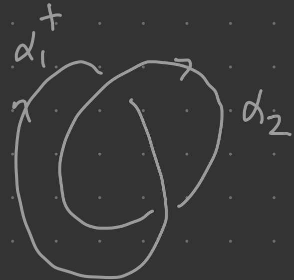
$$\pi_1(K) := \left\{ \begin{matrix} \text{loops in } \mathbb{R}^3 \\ \text{missing } K \end{matrix} \right\} \Big/ \text{"homotopy"}$$

Very powerful group but is difficult to use. (see MATH 444).

Easier example:

Let K be a knot. The K bounds a smoothly embedded "oriented" surface S, called a <u>Seifert surface</u>





push up on surface

linking # is -1

lk # is 0

$$V(K) = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$$

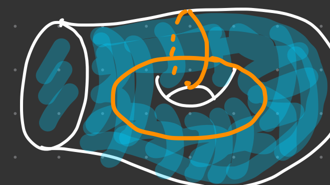
seifert matrix (output)

Q. Is the seifert matrix an invariant?

No, depends on seifert surface chosen

$\emptyset$ matrix

$\sim$



$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

both unknots!

Can use Seifert matrix to get a knot invariant – the Alexander polynomial.

Define:

$$\Delta(K) = \det\left(V(K) - t\,V(K)^T\right)$$

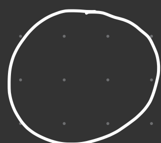
Ex: $V = V(K) = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$ $\quad V - tV^T = \begin{pmatrix} -1+t & -1 \\ t & -1+t \end{pmatrix}$

$$\Delta(K) = \det(V - tV^T) = (t-1)^2 + t$$

$$= t^2 - t + 1$$

[Note: K = trefoil in this example]

This gives more info than $c(K)$ and can be extracted in many ways, including from $\pi_1(K)$, the fundamental group.



$\Delta = 1$

$\Delta = t^2 - t + 1$

$\Delta = t^2 - 3t + 1$

$\Rightarrow$ all different knots

Can also compute via diagrams

L_+ $\quad$ L_- $\quad$ L_0

$$\Delta(O) = 1$$

$$\Delta(L_+) - \Delta(L_-) = (t + t^{-1})\,\Delta(L_0)$$

due to J. Conway

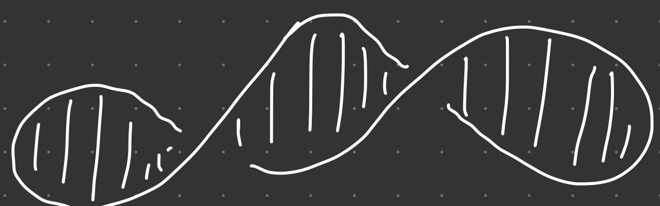
What about 4-dimensions?

4-dimensions in the most difficult and most interesting dimension!

- try to push low-dimensional techniques up (e.g. cut and paste, Heegaard diagrams/trisections, Kirby diagrams)
- try to push high dimensional surgery techniques.

For 4D, we are interested in knots up to "Concordance" instead of isotopy.

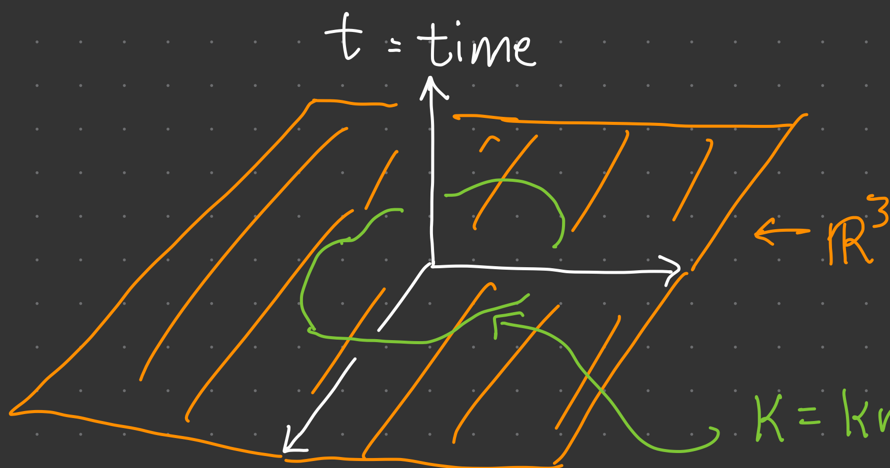
<u>3D</u>: unknot in the only knot that bounds a seifert surface that is a disk (no "holes")

 = 

Start with a knot in $\mathbb{R}^3 \subseteq \mathbb{R}^4$

$(x,y,z) \rightarrow (x,y,z,0)$

the $t=0$ plane
(time)



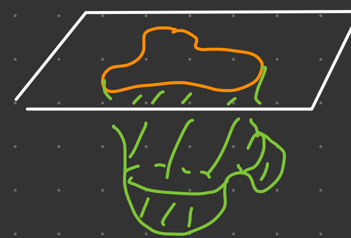
K = knot in $\mathbb{R}^3$

What kind of surface can it bounds in $\mathbb{R}^4_+$

- Take Seifert surface in $\mathbb{R}^3$ in push interior into $\mathbb{R}^4$

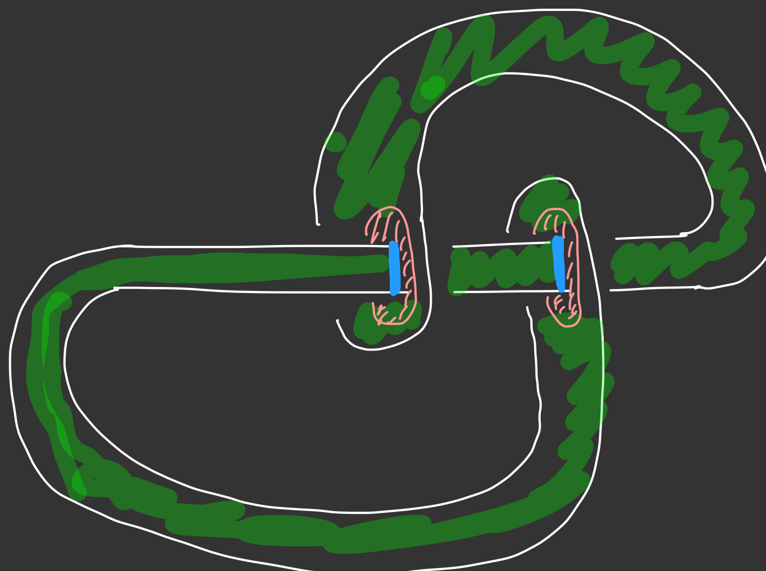
- can get simpler surfaces in theory.

<u>Def:</u> A knot is slice* if it bounds a smooth disk in $\mathbb{R}^4_+$.

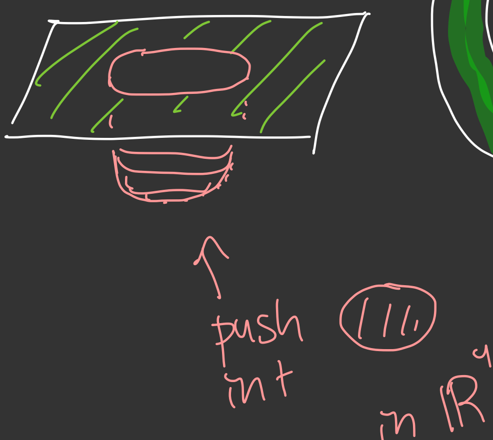


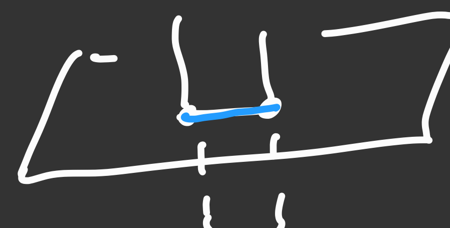
* = two <u>different</u> categories $\nearrow$ smooth $\searrow$ locally flat

Ex 1: (A ribbon knot)



Stevedore knot is slice (does not bound disk in $\mathbb{R}^3$)



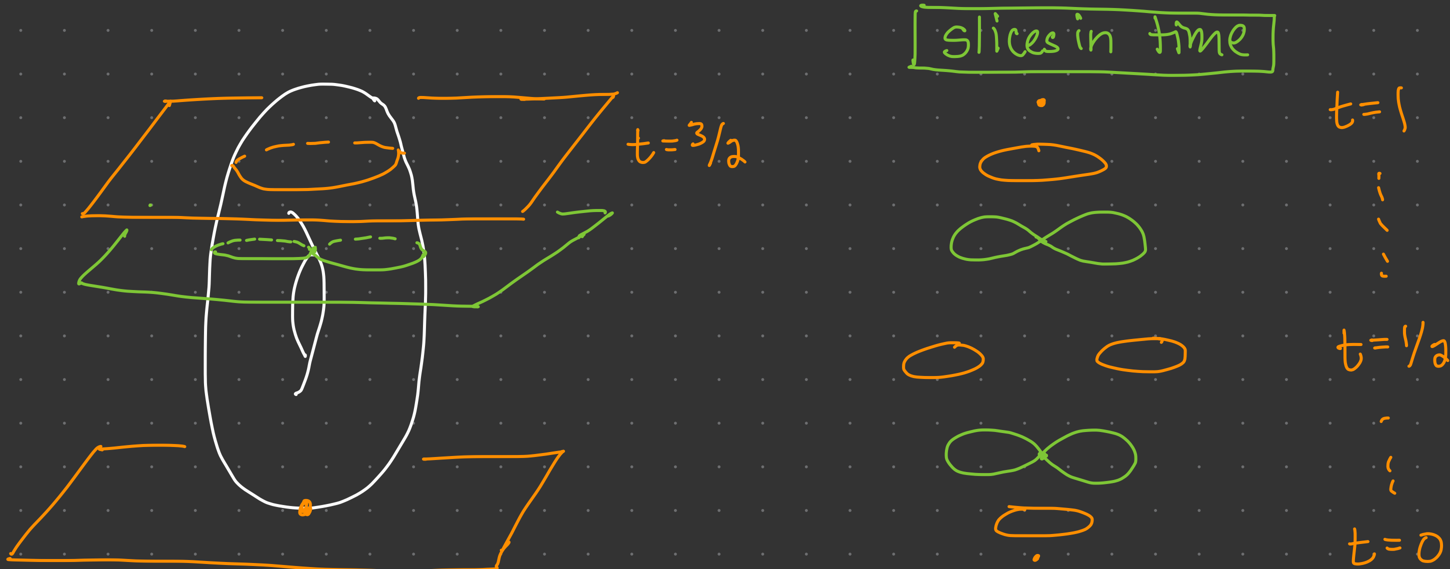


Lemma: A ribbon knot is always slice.

Slice-Ribbon Conjecture: Is every slice knot ribbon

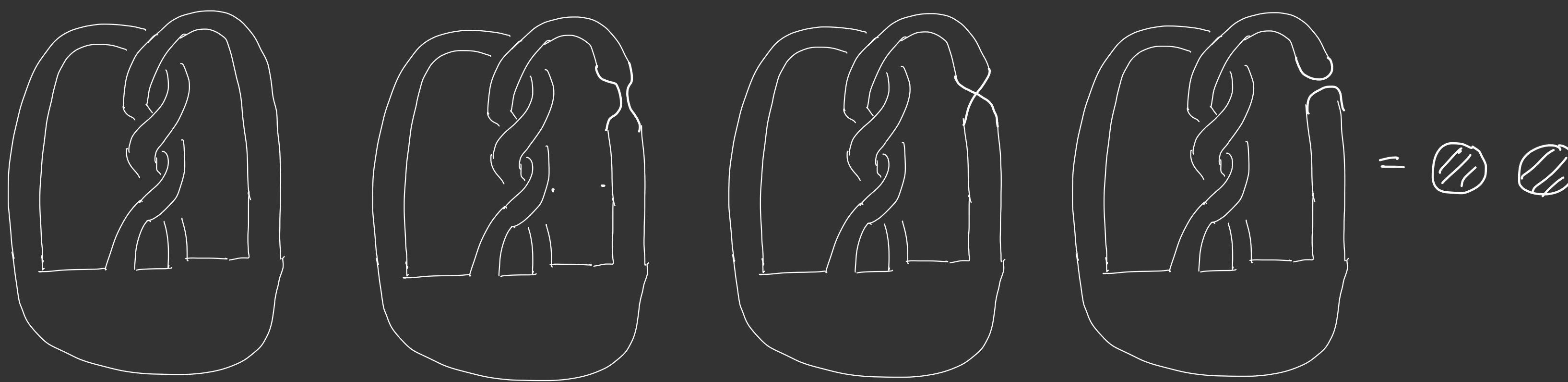
Open conjecture - if you solve, you will be famous.

Recall level sets of surfaces from calculus

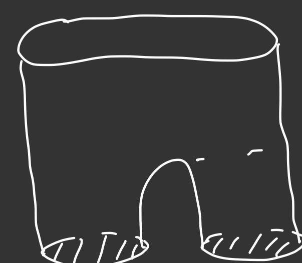


can combine the level sets to produce a surface.

Ex 2: 9_{46} knot is slice (also ribbon)



$t=0$ $\quad$ $t=1/10$ $\quad$ $t=1/5$ $\quad \cdots$ $t=1$



$t=0$

$t=1/10$

$t=1/5$

$t=1$

How to show a particular knot is not slice?

☆ It's HARD ☆

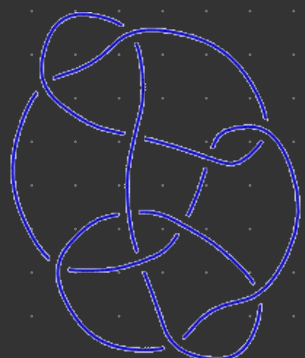
- no algorithm to determine if a given knot is slice (or even ribbon)

open problem!

Thm (Piccirillo, 2018): The Conway knot is not slice.

Annals of Math

smoothly

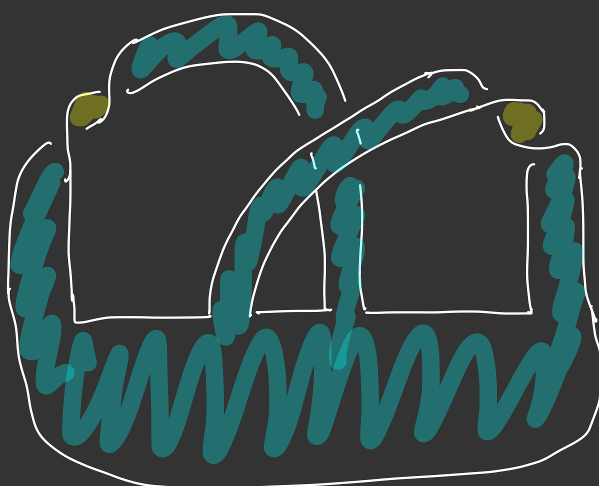


Conway knot

known to (topologically) slice and is a mutant of a slice knot

How to show a knot or <u>not</u> slice.

Start with seifert surface and create seifert matrix V



$$\longrightarrow \quad V = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$$

Want to get special invariant that is "trivial" for slice knots.

<u>Prop</u>: If a knot is slice then its signature is 0.

signature of K, $\sigma(K)$

$$K \longrightarrow V \longrightarrow V+V^T$$

knot   seifert matrix   symmetric matrix

<u>Linear Algebra</u>; $V+V^T$ has real eigenvalues

Define

$$\sigma(K) := (\#\text{pos eigenvalues of } V+V^T) - (\#\text{neg eigenvalues of } V+V^T)$$

Ex:

$$V + V^T = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$$

char poly $= \det(V + V^T - tI)$

$$= \begin{pmatrix} -2-t & -1 \\ -1 & -2-t \end{pmatrix}$$

$$= (2+t)^2 - 1$$

$$= t^2 + 4t + 3$$

roots are eigenvalues:

-2 ± 1 or $-3, -1$

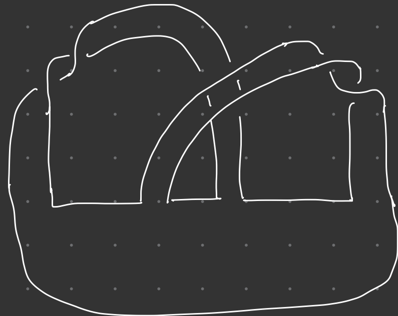
2 neg. eigenvals

$\Longrightarrow$ signature of K is

pos eigenvals

− # neg eigenvals

$= \boxed{-2}$

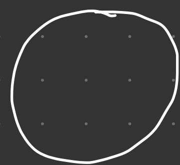
$K =$ 

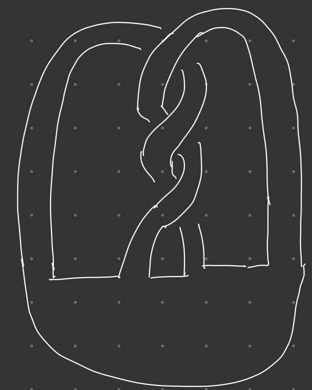
Alexander polynomial of unknot

$\boxed{3D}$ $\Delta(K) = t^2 - t + 1 \neq 1 \implies K$ is not the unknot.

$\boxed{4D}$ $\sigma(K) = -2 \implies K$ is <u>not</u> <u>slice</u>.

What about Alexander polynomial?





$\Delta = 1$

$\Delta = 2t^2 - 5t + 2 = (t-2)(2t-1)$

Both slice $\rightsquigarrow$ cannot obstruct sliceness with full poly.

Prop: If K is slice then $\left|\Delta|_{t=-1}\right|$ is a square.

Ex: 9_{46} ; $\Delta = 2t^2 - 5t + 2 = (t-2)(2t-1)$

$$\Delta|_{t=-1} = (-3)(-3) = 3^2$$

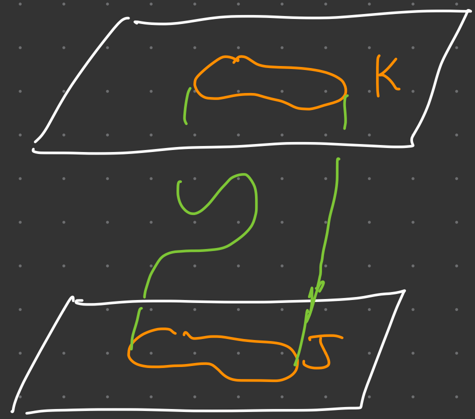
Ex: Trefoil, $\Delta = t^2 - t + 1$

$$\Delta|_{t=-1} = 1+1+1 = 3$$

Concordance

<u>Def:</u> Two knot K, J are concordant if there exists a smooth annulus in $\mathbb{R}^3 \times I$ cobounding K and $-J$

$K \subset \mathbb{R}^3 \times 0$ $\qquad$ $\mathbb{R}^3 \times 1$



<u>Note:</u> slice means concordant to unknot.

Concordance group (abelian)

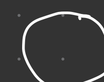
Def: $\mathcal{C} := \{$ equivalence classes of knots with equivalence relation concordance $\}$.

Addition:

 $+$  $:=$ 

connected sum

Zero: unknot

 $+$  $=$ 

Inverse ?

$$K \longrightarrow -K = \text{mirror image}$$

<u>Lemma</u> For any knot $K \# -K$ is ribbon

$\Rightarrow$ equal to 0 in $\mathcal{C}$.

Non-zero elements $\longleftrightarrow$ non slice knots

Ex:  $\neq$ 

$\exists$ elements of ∞-order $\quad T =$ trefoil

consider $\quad nT = T \# T \# \cdots \# T$

$$\sigma(nT) = -2n$$

Ex: Figure eight $= F$ has order two since $F = -F \rightsquigarrow 2F = 0$

HW: Show

$F \qquad\qquad -F$

Open Q: Is there any torsion besides 2-torsion? i.e. $K \# K \# K$ slice but K not slice.

<u>Open question</u> (related to Freedman's surgery conjecture

[from his Fields medal work in '80s]

Is this link (topologically) slice. It's known to not be smoothly slice (A. Levine '14).

whitehead double of Borromean rings?

Thank

you!