

RESEARCH STATEMENT

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1. INTRODUCTION

A variety X over a field K is the vanishing set of a finite number of polynomials whose coefficients are elements of K :

$$X := \{(x_1, \dots, x_n) \in K^n : f_i(x_1, \dots, x_n) = 0 \text{ for } 1 \leq i \leq m\}.$$

Let L be a field extension of K . The set of L -rational points $X(L)$ is the set of n -tuples in L^n who are solutions to the defining polynomials of X :

$$X(L) := \{(x_1, \dots, x_n) \in L^n : f_i(x_1, \dots, x_n) = 0 \text{ for } 1 \leq i \leq m\}.$$

For example, let X be the variety over \mathbb{Q} defined by the polynomial $x^2 + 1 = 0$. Then $X(\mathbb{Q}) = \emptyset$ while $X(\mathbb{C}) = \{i, -i\}$.

“Does a variety X defined over a field K contain K -rational points, and if so, how are these points distributed on X ?” This is a classical question when K is a number field. One approach to this question is to determine whether or not the Local-to-Global principle holds. If X has a point in every completion K_ν of K , i.e. ‘local’ points everywhere, does X contain a K -rational point, i.e. ‘global’ point? For example, a variety defined over \mathbb{Q} has local points everywhere if it has points in $\mathbb{R} = \mathbb{Q}_\infty$ and in \mathbb{Q}_p for all primes p :

$$X(\mathbb{Q}_p) \neq \emptyset \text{ for all } p \text{ prime and } \infty \Rightarrow X(\mathbb{Q}) \neq \emptyset.$$

A similar approach helps analyze the distribution of the K -rational points on X . We say that X satisfies *weak approximation* if for any finite set of places of K and points of X over these places, there exists a K -rational point of X which is arbitrarily close to each of these points. In other words, X is said to satisfy weak approximation if $X(K)$ is dense in $\prod_\nu X(K_\nu)$.

In my research, I study varieties defined over the function field of a curve instead of a number field. Let k be an algebraically closed field of characteristic zero, B a smooth projective curve over k with function field $K = k(B)$. Let X be a smooth, proper variety over K and $\pi : \mathcal{X} \rightarrow B$ a model of X ; i.e., \mathcal{X} is an algebraic space, flat and proper over B , with generic fiber X . In this context, K -rational points of X correspond to sections of π . Weak approximation corresponds to finding sections of π which pass through prescribed points in finitely many fibers with finitely prescribed Taylor series.

The Tseng-Lang theorem states that over the function field of a smooth projective curve, a homogeneous polynomial has a nontrivial solution provided the number of variables is greater than the degree of the polynomial. If X in the preceding paragraph is a hypersurface of degree d in \mathbb{P}_k^n with $d \leq n$, then π admits a section. In 2001 Graber, Harris, and Starr [2] proved the existence of a section for varieties

other than hypersurfaces. If $\pi : \mathcal{X} \rightarrow B$ is a proper morphism of complex varieties with rationally connected general fiber and B a smooth curve, then π has a section. A smooth variety X is said to be *rationally connected* if there is a dense open subset of X in which any two points can be connected by a proper rational curve. Kollár, Miyaoka, and Mori [8] proved that a hypersurface of degree d in \mathbb{P}_k^n is rationally connected if and only if $d \leq n$.

A decade before the Graber, Harris, Starr result Kollár, Miyaoka, and Mori [8] showed that if $\pi : \mathcal{X} \rightarrow B$ is as above and admits a section, then there is a section of π which passes through a finite number of prescribed points in the smooth fibers. They proved that given a finite collection of points $\{b_i\}_{i \in I} \in B$ such that each fiber \mathcal{X}_{b_i} is smooth, and given points $x_i \in \mathcal{X}_{b_i}$, there exists a section, $s : B \rightarrow \mathcal{X}$, such that $s(b_i) = x_i$, for each $i \in I$. In 2004, Hassett and Tschinkel [4] extended this result to show that there exist sections with prescribed tangencies over a finite number of points of B as long as the fibers above these points are smooth. In other words, they proved that a smooth proper rationally connected variety X over $k(B)$ satisfies weak approximation for places of good reduction.

It is conjectured that the above results are true even if the fibers above the points $\{b_i\}_{i \in I} \in B$ are not smooth, as long as the points $x_i \in \mathcal{X}_{b_i}$ are in the smooth locus of the fibers. That is, a smooth rationally connected variety X defined over K satisfies weak approximation even at places of bad reduction.

This conjecture is true if the general fiber is a smooth rationally connected curve. The conjecture is not yet known to be true for all cases where the general fiber is a smooth rationally connected surface. All geometrically rationally connected surfaces are geometrically rational surfaces, and rational surfaces defined over the function field of a curve have been classified. They fall into two categories, del Pezzo surfaces and conic bundles over \mathbb{P}_k^1 . A *del Pezzo surface* of degree $1 \leq d \leq 9$ is the blow-up of the projective plane at $9 - d$ points in general position. Hence, the geometry of the surface becomes more complicated as the degree decreases. The conjecture is known in the following cases:

- conic bundles over \mathbb{P}_k^1 [1],
- del Pezzo surfaces of degree $d \geq 4$ [1],
- degree 3 del Pezzo surfaces, i.e. cubic surfaces, provided they have square-free discriminant [3].

My thesis proves the conjecture for ‘generic’ degree 2 del Pezzo surfaces.

2. CURRENT RESEARCH

In 2005 Hassett and Tschinkel [3] showed that if the fibers of $\pi : \mathcal{X} \rightarrow B$ are strongly rationally connected and π has a section, then sections of π satisfy weak approximation. A smooth rationally connected variety Y is *strongly rationally connected* if any two points in Y can be connected by a proper rational curve. Keel and McKernan [6] proved that the smooth locus of a del Pezzo surface with rational double points is rationally connected. Hassett and Tschinkel [3] conjectured that the smooth locus of a del Pezzo with rational double points is strongly rationally connected. They proved that the conjecture is true when the degree of the del

Pezzo surface is 3 [3]. In my thesis, I prove this conjecture for degree 2 del Pezzo surfaces.

Theorem 1. *The smooth locus of a degree 2 del Pezzo surface with rational double points is strongly rationally connected.*

Theorem 2. *Let X be a smooth degree 2 del Pezzo surface over K and $\pi : \mathcal{X} \rightarrow B$ a model whose singular fibers are degree 2 del Pezzo surfaces with rational double points. Then sections of $\pi : \mathcal{X} \rightarrow B$ satisfy weak approximation.*

Corollary 3. *A smooth degree two del Pezzo surface with square-free discriminant satisfies weak approximation.*

In order to prove Theorem 1, I produce rational curves in the smooth locus of the surface which connect any given point in the smooth locus to a general point. The following proposition reduces this to producing a free curve through any smooth point. (A rational curve $f : \mathbb{P}^1 \rightarrow Y$, nonconstant, is *free* if $f^*\mathcal{T}_Y \simeq \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2)$ where each $a_i \geq 0$.)

Proposition 4. *A smooth rationally connected surface Y is strongly rationally connected if and only if for each point $y \in Y$, there exists a free rational curve containing y .*

A degree 2 del Pezzo surface with rational double points is a double cover of the projective plane branched over a quartic plane curve Q with ADE singularities [5]:

$$\begin{aligned} A_n : & \quad x^{n+1} + y^2 = 0 & n \geq 1 \\ D_n : & \quad x(x^{n-2} + y^2) = 0 & n \geq 4 \\ E_6 : & \quad x^4 + y^3 = 0 \\ E_7 : & \quad y(x^3 + y^2) \\ E_8 : & \quad x^5 + y^3. \end{aligned}$$

Let $f : Y \rightarrow \mathbb{P}^2$ be the double cover of the plane branched over a quartic curve Q . Given any smooth point $y \in Y$, I find a curve $C \subset \mathbb{P}^2$ which passes through $f(y)$ and lifts to a free rational curve $\tilde{C} \subset Y^{sm}$ containing the point y . In order for \tilde{C} to be rational, C must intersect Q in a certain way, depending on the degree of C . The prescribed intersection data necessary for \tilde{C} to be rational has led to a detailed analysis of quartic plane curves with ADE singularities. For example, a quartic curve with six A_1 singularities is the intersection of four lines in general position. One curve C which lifts to a rational curve is a conic that is tangent to three of the lines at smooth points and intersects the other line at two distinct smooth points.

3. FUTURE WORK

3.1. Log del Pezzo Surfaces. We say that a normal projective surface X is a *log del Pezzo surface* if X has at worst quotient singularities and if the anticanonical divisor $-K_X$ of X is ample. In my thesis, I prove the strong rational connectivity of degree 2 log del Pezzo surfaces with only rational double points. Keel and

McKernan [6] have shown that the smooth locus of any log del Pezzo is rationally connected.

Question. *Is the smooth locus of a degree 2 log del Pezzo surface strongly rationally connected?*

The strong rational connectivity of the smooth locus of cubic surfaces with arbitrary quotient singularities is also not known. Proving strong rational connectivity for a larger class of quotient singularities would imply that weak approximation holds for more smooth degree 2 and 3 del Pezzo surfaces.

3.2. Degree 1 del Pezzo Surfaces. As noted in the introduction, at least partial approximation results are known for all rational surfaces except for the degree 1 del Pezzo surfaces. A del Pezzo surface of degree 1 with rational double points can be realized as a hypersurface of degree 6 in the weighted projective space $\mathbb{P}(1, 1, 2, 3)$.

Question. *Is the smooth locus of a degree 1 log del Pezzo surface strongly rationally connected?*

Question. *Does weak approximation hold for degree 1 del Pezzo surfaces?*

3.3. Quartic Threefolds. The only rationally connected hypersurfaces in \mathbb{P}_k^4 are linear, quadric, cubic, and quartic. Weak approximation is known to hold for all hyperplanes, all quartic hypersurfaces, cubic hypersurfaces of dimension at least 5, and quartic hypersurfaces of dimension at least 83 [10]. Recently de Jong and Starr proved that weak approximation holds for hyperplanes of degree d in \mathbb{P}_k^n when

$$n \geq \binom{d^2 + d - 1}{d - 1} + d^2 - 1.$$

In high degrees, this is sharper than the Hassett and Tschinkel bound, but in degrees 3 and 4, $n \geq 63$ and 984, respectively.

Thus, proving weak approximation for cubic and quartic threefolds is still an open problem. Zhang [9] has shown that such hypersurfaces having ordinary singularities are rationally connected. We say that a variety has *ordinary singularities* if it has a finite number of isolated singularities, each étale-locally isomorphic to a cone over a smooth quadric hypersurface.

Question. *Is the smooth locus of a quartic threefold with ordinary singularities strongly rationally connected?*

Suppose that the answer to this question is yes. Then, a smooth quartic threefold, X , with square-free discriminant satisfies weak approximation over B . The equivalent statement for cubic threefolds is true [10].

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