ARITHMETIC OF DEL PEZZO AND K3 SURFACES: EXERCISES II

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1. Del Pezzo surfaces II

(1) Let X be a del Pezzo surface of degree 8 over a field k such that $X^{\mathrm{s}} \cong \mathbb{P}^1_{k^{\mathrm{s}}} \times \mathbb{P}^1_{k^{\mathrm{s}}}$. Note that $\operatorname{Pic} X^{\mathrm{s}} \cong \mathbb{Z}L_1 \oplus \mathbb{Z}L_2$, where L_1 and L_2 are classes representing the two rulings on $\mathbb{P}^1_{k^{\mathrm{s}}} \times \mathbb{P}^1_{k^{\mathrm{s}}}$. We saw in lecture that if $X(k) \neq \emptyset$, or if k is a number field and $X(\mathbb{A}) \neq \emptyset$, then $L_1 + L_2$ is linearly equivalent to a divisor D defined over k.

(a) Use the Riemann-Roch theorem for surfaces and Kodaira vanishing to show that

$$h^0(X, \mathscr{O}_X(D)) = 4.$$

(b) Note that D is a very ample divisor, so the linear system $|D|$ gives an embedding $X \hookrightarrow \mathbb{P}^3_k$. Show that the image is a quadric surface.

(c) Conversely, show that a smooth quadric surface in $\mathbb{P}^3_k$ is a del Pezzo surface of degree 8.

(2) Let X be a del Pezzo surface of degree 6 over a field k. Show that a point $P \in X(k)$ can lie on at most two exceptional curves. Use this to show that if $P \in X(k)$ lies on at least one exceptional curve then X is not k-minimal, i.e., X contains some Galois-stable set of exceptional curves that are pairwise disjoint which can be contracted over k to give a del Pezzo surface of higher degree.

(3) Let $X \subset \mathbb{P}^6_k$ be a del Pezzo surface of degree 6 over a global field k. In the course of proving these surfaces satisfy the Hasse principle, we showed that if X has local point everywhere, then it contains k-points P_1 and P_2 of degrees 2 and 3, respectively. We then argued that if $H \subset \mathbb{P}^6_k$ is a 4-dimensional linear space, then $X \cap H$ has degree 6 (provided $\dim(X \cap H) = 0$) and X must contain a rational point (because P_1 and P_2 account for 5 of the geometric points of $X \cap H$, and the remaining point must be Galois stable). What happens if $\dim(X \cap H) > 0$? Can you still produce a k-point on X?

(4) Here we give an example of a nice variety X over a field k such that the natural map

$$\operatorname{Pic} X \to \left(\operatorname{Pic} X^{\mathrm{s}}\right)^{\operatorname{Gal}(k^{\mathrm{s}}/k)}$$

is not surjective. Let X be a cubic surface containing a Galois-stable set $S = \{e_1, \ldots, e_6\}$ of 6 exceptional curves that do not pairwise intersect. Since the action of Galois preserves the canonical class, we know that

$$-K_X + e_1 + \cdots + e_6 = 3\ell$$

is Galois-stable. Blowing down S we obtain a del Pezzo surface of degree 9, i.e., a Severi-Brauer surface Y. Show that the class $\ell \in \left(\operatorname{Pic} X^{\mathrm{s}}\right)^{\operatorname{Gal}(k^{\mathrm{s}}/k)}$ comes from $\operatorname{Pic} X$ if and only if $Y \cong \mathbb{P}^2_k$.